

# ENDOSCOPIC LIFTING OF SIMPLE SUPERCUSPIDAL REPRESENTATIONS OF $U_{E/F}(N)$ TO $GL_N(E)$

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ABSTRACT. We compute the characters of simple supercuspidal representations of twisted  $GL_N(E)$  and standard  $U_{E/F}(N)$  for a quadratic unramified extension  $E/F$  of  $p$ -adic fields. Comparing them by the endoscopic character relation, we determine the liftings of simple supercuspidal representations of  $U_{E/F}(N)$  to  $GL_N(E)$ , under the assumption that  $p$  is not equal to 2.

## 1. INTRODUCTION

Let  $G$  be a connected reductive group over a  $p$ -adic field  $F$ . Let  $\Pi(G)$  be the set of equivalence classes of irreducible smooth representations of  $G(F)$ , and  $\Phi(G)$  the set of equivalence classes of  $L$ -parameters of  $G$ . Here an  $L$ -parameter of  $G$  is a homomorphism from the Weil-Deligne group  $W_F \times SL_2(\mathbb{C})$  to the  $L$ -group  ${}^L G = \widehat{G} \rtimes W_F$  of  $G$ . Then the conjectural local Langlands correspondence predicts that there exists a natural map from  $\Pi(G)$  to  $\Phi(G)$  with finite fibers ( $L$ -packets).

For  $G = GL_n$ , it was established by Harris-Taylor [HT01] and Henniart [Hen00]. In this case, the map from  $\Pi(G)$  to  $\Phi(G)$  is bijective.

For quasi-split classical groups, the correspondence was established recently by Arthur [Art13] and Mok [Mok15], under the assumption of the stabilization of twisted trace formulas for general linear groups and even orthogonal groups. They characterized the  $L$ -packets for  $L$ -parameters of those groups via the *endoscopic character relation*, and constructed them in consequence of the comparison of stable trace formulas.

We explain the endoscopic character relation. Let  $G$  and  $H$  be quasi-split classical groups over  $F$ . We assume that  $H$  is an endoscopic group for  $G$ , hence we have an  $L$ -embedding  $\iota: {}^L H \hookrightarrow {}^L G$ . By composing this  $L$ -embedding  $\iota$ , we can regard an  $L$ -parameter  $\phi'$  of  $H$  as an  $L$ -parameter  $\phi$  of  $G$ . Then  $\phi$  and  $\phi'$  define  $L$ -packets of  $G$  and  $H$ , respectively.

$$\begin{array}{ccc}
 \Pi(G) \supset \Pi_\phi & \xrightarrow{\text{LLC for } G} & W_F \times SL_2(\mathbb{C}) \xrightarrow{\phi} {}^L G \\
 & & \searrow \phi' \quad \uparrow \iota \\
 \Pi(H) \supset \Pi_{\phi'} & \xrightarrow{\text{LLC for } H} & {}^L H
 \end{array}$$

In this situation, the  $L$ -packet  $\Pi_\phi$  of  $G$  is called the *lifting* of the  $L$ -packet  $\Pi_{\phi'}$  of  $H$ . Then the endoscopic character relation is an equality between the characters of representations in  $\Pi_\phi$  and those of  $\Pi_{\phi'}$ .

Therefore the local Langlands correspondence for  $H$  is reduced to describing that for  $G$  and determining the liftings of  $L$ -packets of  $H$  to  $G$ . For this, it is important to compute the characters of representations.

In our past paper [Oi16], we considered this problem for *simple supercuspidal* representations in the case of  $(G, H) = (\mathrm{GL}_{2n}, \mathrm{SO}_{2n+1})$ . These representations are supercuspidal representations which were introduced in [GR10] and [RY14], and their  $L$ -parameters have been described explicitly in the cases of general linear groups, by the works [BH05] and [IT15]. In [Oi16], we proved that a simple supercuspidal representation of  $\mathrm{SO}_{2n+1}(F)$  constitutes an  $L$ -packet and that its lifting to  $\mathrm{GL}_{2n}(F)$  is again simple supercuspidal.

In this paper, we consider this lifting problem for simple supercuspidal representations in the case of  $(G, H) = (\mathrm{Res}_{E/F}(\mathrm{GL}_N), \mathrm{U}_{E/F}(N))$ , where  $E/F$  is a quadratic unramified extension of  $p$ -adic fields and  $\mathrm{U}_{E/F}(N)$  is the quasi-split unitary group in  $N$  variables. The unitary group  $\mathrm{U}_{E/F}(N)$  is a twisted endoscopic group for  $\mathrm{Res}_{E/F}(\mathrm{GL}_N)$  and there exist two kinds of  $L$ -embeddings  $\{\xi_\kappa\}_{\kappa \in \{\pm 1\}}$  from  ${}^L\mathrm{U}_{E/F}(N)$  to  ${}^L\mathrm{Res}_{E/F}(\mathrm{GL}_N)$  (see Section 4.1 for the details). We determine the liftings of simple supercuspidal representations of  $\mathrm{U}_{E/F}(N)(F)$  to  $\mathrm{Res}_{E/F}(\mathrm{GL}_N)(F) = \mathrm{GL}_N(E)$  via these two kinds of  $L$ -embeddings.

To state our main result, we explain some notations. For simplicity, we consider the case where  $N = 2n$  here. Let  $k_E$  and  $k_F$  be the residue fields of  $E$  and  $F$ , respectively. We write  $(k_E^\times)^*$  and  $\mathrm{U}(1)^*$  for the set of characters on  $k_E^\times$  and  $k_E^\times/k_F^\times$ , respectively. Roughly speaking, simple supercuspidal representations can be obtained by the compact induction of *affine generic* characters (see Section 2.2 for the details). We can parametrize the equivalence classes of such characters in a non-canonical way. For  $\mathrm{U}_{E/F}(2n)(F)$ , they can be parametrized by the finite set  $k_F^\times \times \mathrm{U}(1)^*$ . On the other hand, for  $\mathrm{GL}_{2n}(E)$ , they can be parametrized by the finite set  $k_E^\times \times \mathbb{C}^\times \times (k_E^\times)^*$ . By using these parametrization, for  $(b, \omega') \in k_F^\times \times \mathrm{U}(1)^*$  (resp.  $(a, \zeta, \omega) \in k_E^\times \times \mathbb{C}^\times \times (k_E^\times)^*$ ), we denote the corresponding representations by  $\pi'_{b, \omega'}$  (resp.  $\pi_{a, \zeta, \omega}$ ). Then our main theorem is stated as follows:

**Theorem 1.1** (Even case, Theorem 5.13). *We assume  $p \neq 2$ . Let  $b \in k_F^\times$ ,  $\omega' \in \mathrm{U}(1)^*$ , and  $\kappa \in \{\pm 1\}$ .*

- (1) *The  $L$ -packet containing the simple supercuspidal representation  $\pi'_{b, \omega'}$  of  $\mathrm{U}_{E/F}(2n)(F)$  is a singleton. In particular, the character of  $\pi'_{b, \omega'}$  is stable.*
- (2) *The lifting of the simple supercuspidal representation  $\pi'_{b, \omega'}$  of  $\mathrm{U}_{E/F}(2n)(F)$  to  $\mathrm{GL}_{2n}(E)$  via the  $L$ -embedding  $\xi_\kappa$  is again simple supercuspidal, and given by  $\pi_{b, -\kappa\omega'(\epsilon), \omega'}$ , where  $\epsilon \in k_E^\times$  such that  $\mathrm{Tr}_{k_E/k_F}(\epsilon) = 0$ .*

We remark that we will show similar results in the odd case (Theorem 6.13).

We explain the outline of our proof. Our strategy is basically the same as in [Oi16]. That is, we consider the converse direction.

We first take the simple supercuspidal representation  $\pi = \pi_{b, -\kappa\omega'(\epsilon), \omega'}$  of  $\mathrm{GL}_{2n}(E)$ . We can easily check this representation is conjugate self-dual. Thus the  $L$ -parameter  $\phi$  of  $\pi$  factors through either  $\xi_{+1}$  or  $\xi_{-1}$ . To show that  $\phi$  factors through  $\xi_\kappa$ , we compute the twisted character of  $\pi$  for special elements, which we call *affine generic* elements. By using the twisted character formula for supercuspidal representations, we write these character values explicitly in terms of generalized Kloosterman sums. Combining the endoscopic character relation with some properties of generalized

Kloosterman sums, we can prove that  $\phi$  factors through  $\xi_\kappa$ . Then we get the  $L$ -packet of  $U_{E/F}(2n)$ , and know this is a singleton consisting of a supercuspidal representation by results in [Mok15] and [Mœg07].

The key point of the proof is to show that this unique representation  $\pi'$  in the  $L$ -packet is given by  $\pi'_{b,\omega'}$ . To prove this, we again use the endoscopic character relation. By the endoscopic character relation, we can express the character of  $\pi'$  in terms of the twisted character of  $\pi$ , which is already computed. From this, we can show that  $\pi'$  is either simple supercuspidal or depth-zero supercuspidal. To eliminate the possibility that  $\pi'$  is depth-zero supercuspidal, we next compute the character of depth-zero supercuspidal representations, and compare them. Once we know that  $\pi'$  is simple supercuspidal, we can show that  $\pi' = \pi'_{b,\omega'}$  easily by computing the characters of simple supercuspidal representations of  $U_{E/F}(2n)(F)$  and considering the Fourier transform of Kloosterman sums, and this completes the proof.

We explain the organization of this paper. In Section 2, we review some fundamental properties about Iwahori subgroups and simple supercuspidal representations. In addition, we introduce the notion of affine genericity for elements in Iwahori subgroups, which will play important roles in a comparison of characters. In Section 3, we compute the characters of simple supercuspidal representations of twisted  $GL_N(E)$  and standard  $U_{E/F}(N)(F)$  for affine generic elements. We treat the even case and the odd case separately. In Section 4, we investigate the norm correspondence for  $GL_N(E)$  and  $U_{E/F}(N)(F)$ . The norm correspondence is used to formulate the endoscopic character relation. We determine norms of affine generic elements and compute their transfer factors. In Section 5, we first recall the endoscopic character relation in [Mok15]. Then we determine the liftings of simple supercuspidal representations by combining it with the results in Sections 3 and 4. In Appendix A, we list some properties about the Fourier transform of Kloosterman sums.

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**Notation.** Let  $p$  be an odd prime number. For a  $p$ -adic field  $F$ , we denote its ring of integers, its maximal ideal, and its residue field by  $\mathcal{O}_F$ ,  $\mathfrak{p}_F$ , and  $k_F$ , respectively. For  $x \in \mathcal{O}_F$ , let  $\bar{x}$  denote the image of  $x$  in  $k_F$ .

We fix a quadratic unramified extension  $E/F$  of  $p$ -adic fields. We write  $\Gamma$  for the absolute Galois group  $\text{Gal}(\overline{F}/F)$  of  $F$ , and  $c$  for the Galois conjugation of  $E/F$ . We fix a uniformizer  $\varpi \in F^\times$ . For simplicity, we denote  $\text{Nr}_{k_E/k_F}$  and  $\text{Tr}_{k_E/k_F}$  by  $\text{Nr}$  and  $\text{Tr}$ , respectively. We fix a non-square element  $\epsilon_F \in k_F^\times$  and its square root  $\epsilon \in k_E^\times$ . We denote the set  $\{z \in k_E^\times \mid \text{Nr}(z) = 1\}$  by  $U(1)$ .

For an algebraic group  $T$ , we write  $X^*(T)$  for its character group and  $X_*(T)$  for its cocharacter group.

## 2. SIMPLE SUPERCUSPIDAL REPRESENTATIONS

**2.1. Iwahori subgroups.** Let  $G$  be a connected reductive group over  $F$ , and  $Z$  the center of  $G$ . For simplicity, we often identify  $Z$  with the set of  $F$ -rational points  $Z(F)$ . Let  $S$  be a maximal  $F$ -split torus in  $G$ ,  $N_S$  the normalizer of  $S$  in  $G$ , and  $Z_S$  the centralizer of  $S$  in  $G$ . We write  $\mathcal{A}(G, S) \cong X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$  for the apartment of  $S$ .

We denote the set of roots of  $S$  in  $G$  by  $\Phi$ , and the set of affine roots by  $\Psi$ . For each root  $a \in \Phi$ , we denote by  $U_a$  the corresponding root subgroup. For an affine function  $\alpha$  on  $\mathcal{A}(G, S)$  whose gradient  $\dot{\alpha}$  belongs to  $\Phi$ , we denote by  $U_\alpha$  its corresponding subgroup of  $U_{\dot{\alpha}}(F)$  (see [Tit79, 1.4]). For an affine function  $\alpha$  on  $\mathcal{A}(G, S)$  whose gradient  $\dot{\alpha}$  does not belong to  $\Phi$ , we set  $U_\alpha = 1$ . We remark that if  $\dot{\alpha} \in \Phi$ , then we have the following chain of subgroups:

$$U_{2\alpha} \subset U_\alpha \subset U_{\dot{\alpha}}(F).$$

We denote by  $\bar{U}_\alpha$  the quotient  $U_\alpha/U_{\alpha+\varepsilon}$  for a sufficiently small number  $\varepsilon > 0$ .

We fix an alcove  $C$  in  $\mathcal{A}(G, S)$  of  $S$  in  $G(F)$ . This determines an affine root basis  $\Pi$  of  $\Psi$  and the set  $\Psi^+$  of positive affine roots. We set the Iwahori subgroup associated to  $C$  and its subgroups as follows:

$$\begin{aligned} I &:= \langle (Z_S)_0, U_\alpha \mid \alpha \in \Psi^+ \rangle, \\ I^+ &:= \langle (Z_S)_1, U_\alpha \mid \alpha \in \Psi^+ \rangle, \text{ and} \\ I^{++} &:= \langle (Z_S)_1, U_\alpha \mid \alpha \in \Psi^+ \setminus \Pi \rangle, \end{aligned}$$

where  $(Z_S)_0$  is the maximal compact subgroup of  $Z_S(F)$ , and  $(Z_S)_1$  is the pro-unipotent radical of  $(Z_S)_0$ . These groups are the first three steps of the Moy-Prasad filtration of the Iwahori subgroup  $I$  associated to the barycenter of the alcove  $C$  (see [RY14, Section 2.6]). We note that  $I^+$  and  $I^{++}$  are normal in  $I$ , and the graded quotients are finite abelian groups.

We define the subgroups of  $Z_S(F)$  by

$$\begin{aligned} Z_S(q) &:= \{t \in Z_S(F) \mid t \text{ has finite order}\} \text{ and} \\ Z(q) &:= \{t \in Z(F) \mid t \text{ has finite order}\}. \end{aligned}$$

These are sets of representatives of  $(Z_S)_0/(Z_S)_1$  and  $(Z \cap (Z_S)_0)/(Z \cap (Z_S)_1)$ .

**Proposition 2.1.** (1) *We have*

$$I/I^+ \cong (Z_S)_0/(Z_S)_1 \cong Z_S(q).$$

(2) *The quotient  $V := I^+/I^{++}$  is a finite-dimensional  $k_F$ -vector space and  $S(k_F)$  acts on it. Moreover we have*

$$I^+/I^{++} \cong \bigoplus_{\alpha \in \Pi} V(\dot{\alpha}).$$

*Here,  $V(\dot{\alpha})$  is the  $\dot{\alpha}$ -isotypic part of  $V$  with respect to the  $S(k_F)$ -action.*

*Proof.* We prove (2). Since the quotient  $I^+/I^{++}$  is abelian, we have the following well-defined map:

$$\bigoplus_{\alpha \in \Pi} U_\alpha \rightarrow I^+/I^{++}.$$

By the definition of  $I^+$  and  $I^{++}$ , this map is surjective.

Let  $\alpha \in \Pi$ , and  $\alpha + r$  the minimal affine root which is greater than  $\alpha$ . Then we have  $\alpha + r > \alpha + \varepsilon > \alpha$ . Since  $\alpha + \varepsilon$  is not an affine root, we have  $\bar{U}_{\alpha+\varepsilon} = \bar{U}_{2(\alpha+\varepsilon)}$ . Thus we have  $U_{\alpha+\varepsilon} = U_{\alpha+r} \cdot U_{2(\alpha+\varepsilon)} \subset U_{\alpha+r} \cdot U_{2\alpha}$ .

On the other hand, since  $\alpha + r$  and  $2\alpha$  are positive and not simple affine roots, the map  $U_\alpha \rightarrow I^+/I^{++}$  factors through  $U_\alpha/(U_{\alpha+r} \cdot U_{2\alpha})$ . Hence it factors through  $\bar{U}_\alpha/\bar{U}_{2\alpha}$ . As  $\bar{U}_\alpha/\bar{U}_{2\alpha}$  is a finite-dimensional  $k_F$ -vector space (see [Tit79, 1.6]) and  $S(k_F)$  acts on it by  $\dot{\alpha}$ , this completes the proof.  $\square$

We denote the image of  $x$  under the map  $I^+ \twoheadrightarrow \bigoplus_{\alpha \in \Pi} V(\dot{\alpha})$  by  $(x_\alpha)_{\alpha \in \Pi}$ , and call each  $x_\alpha$  the *affine simple component* of  $x$ .

**Definition 2.2.** (1) An element  $x \in I^+$  is said to be *affine generic* if  $x_\alpha$  is nonzero for every  $\alpha \in \Pi$ .

(2) A character  $\psi: I^+ \rightarrow \mathbb{C}^\times$  is called *affine generic* if it factors through the quotient  $I^+/I^{++}$  and is nontrivial on  $V(\dot{\alpha})$  for every  $\alpha \in \Pi$ .

**Proposition 2.3** ([Ric13]). *Let  $\widetilde{W}$  be the Iwahori-Weyl group of  $S$  defined by*

$$\widetilde{W} := N_S(F)/(Z_S)_0.$$

*Then the following hold.*

- (1) *We have  $G(F) = IN_S(F)I$ , and the map  $InI \mapsto \dot{n}$  induces a bijection*

$$I \backslash G(F)/I \cong \widetilde{W}.$$

- (2) *We have  $G(F) = I^+N_S(F)I^+$ , and the map  $I^+nI^+ \mapsto \dot{n}$  induces a bijection*

$$I^+ \backslash G(F)/I^+ \cong N_S(F)/(Z_S)_1.$$

- (3) *There exists an exact sequence*

$$1 \rightarrow W_{\text{aff}} \rightarrow \widetilde{W} \xrightarrow{\kappa_G} X^*(Z(\widehat{G})^{I_F})^{\Sigma_F} \rightarrow 1,$$

*where  $W_{\text{aff}}$  is the affine Weyl group of  $S$ ,  $I_F := \text{Gal}(\widehat{F}^{\text{sep}}/\widehat{F}^{\text{ur}})$ ,  $\Sigma_F := \text{Gal}(F^{\text{ur}}/F)$ , and  $\kappa_G$  is the Kottwitz homomorphism defined in [Kot97, Section 7]. Moreover the subgroup  $\widetilde{\Omega} \subset \widetilde{W}$  consisting of the elements normalizing  $I$  maps isomorphically to  $X^*(Z(\widehat{G})^{I_F})^{\Sigma_F}$ , and we have  $\widetilde{W} \cong W_{\text{aff}} \rtimes \widetilde{\Omega}$ .*

We fix a set of representatives  $\Omega \subset N_S(F)$  of  $\widetilde{\Omega} \subset \widetilde{W} = N_S(F)/(Z_S)_0$ . Let  $N_G(I)$  and  $N_G(I^+)$  be the normalizers of  $I$  and  $I^+$  in  $G(F)$ , respectively.

**Lemma 2.4.** *We have*

$$N_G(I) = N_G(I^+) = I\Omega.$$

*Proof.* Since  $I^+$  is the pro-unipotent radical of  $I$ , we have  $N_G(I) \subset N_G(I^+)$ . We prove the other inclusion. Let  $g \in N_G(I^+)$ . It suffices to prove  $I^+gI^+ \subset N_G(I)$ . By Proposition 2.3 (2), we can replace  $g$  with  $n \in N_S(F)$ . As  $nI^+n^{-1} = I^+$  and  $nZ_Sn^{-1} = Z_S$ , we have  $n \in N_G(I)$ . Hence  $N_G(I) = N_G(I^+)$ .

We next prove the second equality. The inclusion  $N_G(I) \supset I\Omega$  follows from the definition of  $\Omega$ . Let  $g \in N_G(I)$ . It suffices to prove  $IgI \subset I\Omega$ . By Proposition 2.3 (1), we may assume  $g \in N_S(F)$ . Then we have  $g \in \Omega(Z_S)_0$  by the definition of  $\Omega$ . Hence  $IgI \subset I\Omega$ .  $\square$

The following lemma is a key to compute characters.

**Lemma 2.5.** *Let  $y \in G(F)$ . If  $y$  satisfies  $yyg^{-1} \in I$  for an affine generic element  $g \in I^+$ , then  $y \in N_G(I) = I\Omega$ .*

*Proof.* Let  $y \in G(F)$  satisfying  $yyg^{-1} \in I$  for an affine generic element  $g \in I^+$ . Since affine genericity is preserved by  $I^+$ -conjugation, any element of  $IyI^+$  satisfies the same condition as  $y$ . Therefore, by Proposition 2.3, we may assume  $y \in N_S(F)$ .

As  $yIy^{-1}$  and  $I$  have the same volume for a Haar measure of  $G$ , we only have to show  $yIy^{-1} \subset I$ .

We recall that the multiplication map

$$\prod_{a \in \Phi_{\text{red}}^+} U_a(F) \times Z_S(F) \times \prod_{a \in \Phi_{\text{red}}^-} U_a(F) \rightarrow G(F)$$

is injective in any order (see [BT84, 2.2.3]), where  $\Phi_{\text{red}}^+$  (resp.  $\Phi_{\text{red}}^-$ ) is the set of non-divisible positive (resp. negative) roots of  $S$ . Moreover this induces a bijection

$$\prod_{a \in \Phi_{\text{red}}^+} U_{a+r_a} \times (Z_S)_0 \times \prod_{a \in \Phi_{\text{red}}^-} U_{a+r_a} \rightarrow I$$

in any order, where  $r_a \in \mathbb{R}$  is the smallest number such that  $a + r_a \geq 0$  with respect to the fixed alcove  $C$  (see [BT72, 6.4.9]). We note that if  $a = \dot{\alpha}$  (resp.  $a = \dot{\alpha}/2$ ) for  $\alpha \in \Pi$ , then  $a + r_a = \alpha$  (resp.  $a + r_a = \alpha/2$ ) since a simple affine root vanishes on some wall of the alcove.

By using this decomposition, we write

$$g = \prod_{a \in \Phi_{\text{red}}^+} x_a \cdot t \cdot \prod_{a \in \Phi_{\text{red}}^-} x_a, \text{ and}$$

$$ygy^{-1} = \prod_{a \in \Phi_{\text{red}}^+} yx_a y^{-1} \cdot yty^{-1} \cdot \prod_{a \in \Phi_{\text{red}}^-} yx_a y^{-1},$$

where  $t \in (Z_S)_0$  and  $x_a \in U_{a+r_a}$  for each  $a \in \Phi_{\text{red}}$ . From Proposition 2.1 and the affine genericity of  $x$ , if  $a = \dot{\alpha}$  (resp.  $a = \dot{\alpha}/2$ ) for  $\alpha \in \Pi$ , then the image of  $x_a \in U_\alpha$  in  $\bar{U}_\alpha$  (resp. the image of  $x_a \in U_{\alpha/2}$  in  $\bar{U}_{\alpha/2}$ ) is not zero (note that  $\bar{U}_{\alpha/2} \cong \bar{U}_\alpha$  since  $\alpha/2$  is not an affine root).

Therefore if the gradient  $a$  of a simple affine root  $\alpha = a + r \in \Pi$  is reduced, then  $\alpha$  is the maximal affine function such that

- its gradient is  $a$  and
- $x_a$  belongs to  $U_\alpha$ .

Hence the affine root  $y\alpha y^{-1} \in \Psi$  is the maximal affine function such that

- its gradient is  $yay^{-1}$  and
- $yx_a y^{-1}$  belongs to  $U_{y\alpha y^{-1}}$ .

Similarly, if the gradient  $2a$  of a simple affine root  $\alpha = 2a + r \in \Pi$  is non-reduced, then  $\alpha$  is the maximal affine function such that

- its gradient is  $2a$  and
- $x_a$  belongs to  $U_{\alpha/2}$ .

Hence the affine root  $y\alpha y^{-1} \in \Psi$  is the maximal affine function such that

- its gradient is  $y(2a)y^{-1}$  and
- $yx_a y^{-1}$  belongs to  $U_{y\alpha y^{-1}/2}$ .

On the other hand, by the above uniqueness of expression and the assumption that  $ygy^{-1} \in I$ ,  $yx_a y^{-1}$  belongs to  $U_{yay^{-1}+r_{yay^{-1}}}$  for every  $a \in \Phi_{\text{red}}$ . For  $\alpha \in \Pi$ , by the maximality of  $y\alpha y^{-1}$ , we get

$$\begin{cases} y\alpha y^{-1} \geq y\dot{\alpha} y^{-1} + r_{y\dot{\alpha} y^{-1}} \geq 0 & (\dot{\alpha} \in \Phi_{\text{red}}) \\ y\alpha y^{-1}/2 \geq y\dot{\alpha} y^{-1}/2 + r_{y\dot{\alpha} y^{-1}/2} \geq 0 & (\dot{\alpha} \notin \Phi_{\text{red}}). \end{cases}$$

Therefore we have  $y\alpha y^{-1} \in \Psi^+$  for every positive affine root  $\alpha \in \Psi^+$ . This implies  $yIy^{-1} \subset I$  (note that we have  $y(Z_S)_0 y^{-1} = (Z_S)_0$  since  $y \in N_S(F)$ ).  $\square$

**2.2. Simple supercuspidal representations.** Let  $\psi$  be a character on  $ZI^+$  such that  $\psi|_{I^+}$  is affine generic. We put

$$N(\psi) := \{n \in N_G(I^+) \mid \psi^n = \psi\},$$

where  $\psi^n$  is a character of  $I^+$  defined by  $\psi^n(g) := \psi(ngn^{-1})$ . This subgroup satisfies  $ZI^+ \subset N(\psi) \subset I\Omega$  by Lemma 2.4. For an irreducible constituent  $\chi$  of  $\text{c-Ind}_{ZI^+}^{N(\psi)} \psi$ , we define

$$\pi_\chi := \text{c-Ind}_{N(\psi)}^{G(F)} \chi.$$

*Remark 2.6.* In this paper, we will consider the cases of  $G = \text{Res}_{E/F}(\text{GL}_N)$  and  $G = \text{U}_{E/F}(N)$ , where  $E/F$  is a quadratic unramified extension of  $p$ -adic fields.

- (1) For  $G = \text{Res}_{E/F}(\text{GL}_N)$ , the quotient  $N(\psi) \twoheadrightarrow N(\psi)/ZI^+$  splits, and  $\text{c-Ind}_{ZI^+}^{N(\psi)} \psi$  decomposes as a direct sum of characters.
- (2) For  $G = \text{U}_{E/F}(N)$ , the group  $\tilde{\Omega}$  is trivial (we will check this in Section 2.4), and  $N(\psi) = ZI^+$ .

**Proposition 2.7** ([RY14, Proposition 2.4] and [GR10, Proposition 9.3]). (1)

*We have a decomposition*

$$\text{c-Ind}_{ZI^+}^{G(F)} \psi \cong \bigoplus_{\chi} \dim(\chi) \cdot \pi_{\chi},$$

*where the sum is over the set of irreducible constituents of  $\text{c-Ind}_{ZI^+}^{N(\psi)} \psi$ .*

- (2) *The representation  $\pi_{\chi}$  is irreducible, hence supercuspidal.*
- (3) *Let  $(\psi', \chi')$  be another pair as above. Then,  $\pi_{\chi}$  and  $\pi_{\chi'}$  are equivalent if and only if  $\psi^n = \psi'$  and  $\chi^n \cong \chi'$  for some  $n \in (Z_S)_0 \Omega$ .*

*Proof.* For  $g \in G(F)$  and a subgroup  $J$  of  $G(F)$ , we set  $J^g := g^{-1}Jg$ .

Since  $ZI^+$  is normal in  $N(\psi)$  and the quotient  $N(\psi)/ZI^+$  is finite, we have

$$\text{c-Ind}_{ZI^+}^{N(\psi)} \psi \cong \bigoplus_{\chi} \dim(\chi) \cdot \chi.$$

Hence it suffices to prove (2) and (3) by the transitivity of compact induction. By Mackey's theorem, we have

$$\text{Hom}_{G(F)}(\pi_{\chi}, \pi_{\chi'}) \cong \bigoplus_{n \in N(\psi) \backslash G(F) / N(\psi')} \text{Hom}_{N(\psi)^n \cap N(\psi')}(\chi^n, \chi').$$

Let  $n \in G(F)$  such that  $\text{Hom}_{N(\psi)^n \cap N(\psi')}(\chi^n, \chi') \neq 0$ . Then we may assume  $n \in N_S(F)$  by Proposition 2.3 (2). Since  $(ZI^+)^n \cap ZI^+ \subset N(\psi)^n \cap N(\psi)$ , we also have  $\text{Hom}_{(ZI^+)^n \cap ZI^+}(\chi^n, \chi') \neq 0$ . As  $\chi^n|_{(ZI^+)^n} = (\psi^n)^{\oplus \dim \chi}$ , and  $\chi'|_{ZI^+} = \psi'^{\oplus \dim \chi'}$ , we have  $\psi^n = \psi'$  on  $(ZI^+)^n \cap ZI^+$ .

We show that the image  $w$  of  $n$  under

$$I^+ \backslash G(F) / I^+ \twoheadrightarrow W_{\text{aff}} \rtimes \tilde{\Omega}$$

lies in  $\tilde{\Omega}$ . We assume that  $w \notin \tilde{\Omega}$ . Then we can take a simple affine root  $\alpha \in \Pi$  such that  $w(\alpha) \in \Psi^+ \setminus \Pi$  (see [GR10, Lemma 9.1]). By the definition of  $I^{++}$ ,  $U_{w(\alpha)}$  is contained in  $I^{++}$ . Hence  $\psi$  is trivial on  $U_{w(\alpha)}$ , and  $\psi^n$  is trivial on  $U_{w(\alpha)}^n = n^{-1}U_{w(\alpha)}n = U_{\alpha}$ . Since  $U_{\alpha} = U_{w(\alpha)}^n \subset (ZI^+)^n \cap ZI^+$ , we have  $\psi^n = \psi'$  on  $U_{\alpha}$ . However  $\psi'$  is nontrivial on  $U_{\alpha}$  since  $\alpha \in \Pi$ . This is a contradiction.

As  $w \in \tilde{\Omega}$ ,  $n$  normalizes  $I$ , hence also normalizes  $I^+$ . Therefore we have  $ZI^+ \subset N(\psi)^n \cap N(\psi')$ , and  $\psi^n = \psi'$ . Since  $N(\psi)^n = N(\psi^n) = N(\psi')$ , we have  $\chi^n \cong \chi'$ .

We finally show the irreducibility. We take  $\chi = \chi'$ . Then we have

$$\text{Hom}_{G(F)}(\pi_{\chi}, \pi_{\chi}) \cong \text{Hom}_{N(\psi)}(\chi, \chi),$$

and the dimension of the right-hand side is one.  $\square$

The irreducible supercuspidal representations  $\pi_{\chi}$  constructed in this way are called *simple supercuspidal* representations of  $G(F)$ .



**2.3. Parametrization: the case of  $G_{E/F}(N)$ .** In this subsection, we consider the case of  $G_{E/F}(N) := \text{Res}_{E/F}(\text{GL}_N)$ . Let  $G := G_{E/F}(N)$ . Then we have  $G(F) = \text{GL}_N(E)$ .

We choose  $S$  to be the subgroup of diagonal matrices:

$$S(F) = \{t = \text{diag}(t_1, \dots, t_N) \in \text{GL}_N(E) \mid t_i \in F^\times\}.$$

This is a maximal  $F$ -split torus of  $G$ , and its centralizer  $Z_S$  is given by

$$Z_S(F) = \{t = \text{diag}(t_1, \dots, t_N) \in \text{GL}_N(E) \mid t_i \in E^\times\}.$$

Then we have

$$\begin{aligned} \Phi &= \{\pm(e_i - e_j) \mid 1 \leq i < j \leq N\}, \text{ and} \\ \Psi &= \{a + r \mid a \in \Phi, r \in \mathbb{Z}\}. \end{aligned}$$

We take the root basis

$$\Delta = \{e_1 - e_2, \dots, e_{N-1} - e_N\}$$

corresponding to the Borel subgroup  $B$  consisting of upper triangular matrices. We let  $C$  be the fundamental alcove of  $\mathcal{A}(G, S)$  (i.e.,  $C$  is contained in the chamber which is defined by  $B$ , and the closure  $\overline{C}$  of  $C$  contains 0). Then the corresponding affine root basis is

$$\Pi = \{e_1 - e_2, \dots, e_{N-1} - e_N, e_N - e_1 + 1\},$$

and the Iwahori subgroup and its filtrations are given by

$$\begin{aligned} I &= \begin{pmatrix} \mathcal{O}_E^\times & & \mathcal{O}_E \\ & \ddots & \\ \mathfrak{p}_E & & \mathcal{O}_E^\times \end{pmatrix}, \quad I^+ = \begin{pmatrix} 1 + \mathfrak{p}_E & & \mathcal{O}_E \\ & \ddots & \\ \mathfrak{p}_E & & 1 + \mathfrak{p}_E \end{pmatrix}, \text{ and} \\ I^{++} &= \begin{pmatrix} 1 + \mathfrak{p}_E & \mathfrak{p}_E & & \mathcal{O}_E \\ & \ddots & \ddots & \\ & & \mathfrak{p}_E & \ddots \\ \mathfrak{p}_E^2 & & & 1 + \mathfrak{p}_E \end{pmatrix}. \end{aligned}$$

For  $x = (x_{ij})_{ij} \in I^+$ , we regard its affine simple components  $(x_\alpha)_\alpha \in \bigoplus_{\alpha \in \Pi} V(\dot{\alpha})$  as an element of  $k_E^{\oplus N}$  by

$$\begin{aligned} I^+/I^{++} &\cong \bigoplus_{\alpha \in \Pi} V(\dot{\alpha}) \cong k_E^{\oplus N} \\ (x_{ij})_{ij} &\mapsto (\overline{x_{12}}, \dots, \overline{x_{N-1,N}}, \overline{x_{N1}\varpi^{-1}}). \end{aligned}$$

For  $a \in k_E^\times$ , we set

$$\varphi_a := \begin{pmatrix} 0 & I_{N-1} \\ \varpi a & 0 \end{pmatrix} \in G(F).$$

Here, we regard  $a$  as an element of  $E^\times$  by the Teichmüller lift. This element satisfies  $\varphi_a^N = \varpi a I_N$ , and we can choose a set of representatives  $\Omega$  of  $\tilde{\Omega}$  to be  $\langle \varphi_a \rangle$ .

We fix a nontrivial additive character  $\psi: k_F \rightarrow \mathbb{C}^\times$ . For  $a \in k_E^\times$ , we define an affine generic character  $\psi_a: I^+ \rightarrow \mathbb{C}^\times$  by

$$\psi_a(x) := \psi \circ \text{Tr} \left( \overline{x_{12}} + \dots + \overline{x_{N-1,N}} + \overline{a x_{N1} \varpi^{-1}} \right) \text{ for } x = (x_{ij})_{ij} \in I^+.$$

Then we have  $N(\psi_a) = ZI^+\langle\varphi_{a^{-1}}\rangle$ . For  $\zeta \in \mathbb{C}^\times$  and a character  $\omega$  on  $k_E^\times$ , let  $\chi_{a,\zeta,\omega}: ZI^+\langle\varphi_{a^{-1}}\rangle \rightarrow \mathbb{C}^\times$  be the character defined by

$$\begin{aligned}\chi_{a,\zeta,\omega}(z) &= \omega(\bar{z}) \text{ for } z \in Z(q), \\ \chi_{a,\zeta,\omega}(x) &= \psi_a(x) \text{ for } x \in I^+, \text{ and} \\ \chi_{a,\zeta,\omega}(\varphi_{a^{-1}}) &= \zeta.\end{aligned}$$

Let  $\pi_{a,\zeta,\omega}$  be the simple supercuspidal representation of  $G_{E/F}(N)(F)$  defined by

$$\pi_{a,\zeta,\omega} := \text{c-Ind}_{ZI^+\langle\varphi_{a^{-1}}\rangle}^{G(F)} \chi_{a,\zeta,\omega}.$$

Then, by Proposition 2.7 (3), we can check that the set

$$\{(a, \zeta, \omega) \mid a \in k_E^\times, \zeta \in \mathbb{C}^\times, \omega: k_E^\times \rightarrow \mathbb{C}^\times\}$$

parametrizes the set of equivalent classes of simple supercuspidal representations of  $G_{E/F}(N)(F)$ .

**2.4. Parametrization: the case of  $U_{E/F}(2n)$ .** In this subsection, we consider the case of

$$U_{E/F}(2n) := \{g \in \text{Res}_{E/F}(\text{GL}_{2n}) \mid {}^t c(g)Jg = J\},$$

with

$$J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & & \ddots & \\ (-1)^{2n-1} & & & \end{pmatrix}.$$

Let  $H := U_{E/F}(2n)$ . Then we have  $U_{E/F}(2n)(F) = \{g \in \text{GL}_{2n}(E) \mid {}^t c(g)Jg = J\}$ . We identify the center  $Z_H$  of  $H$  with  $U_{E/F}(1)$ , where

$$U_{E/F}(1)(F) = \{z \in E^\times \mid zc(z) = 1\}.$$

We choose  $S_H$  to be the following subgroup of diagonal matrices in  $H$ :

$$S_H(F) = \{t = \text{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \mid t_i \in F^\times\}.$$

This is a maximal  $F$ -split torus of  $H$ , and its centralizer  $Z_{S_H}$  is given by

$$Z_{S_H}(F) = \{t = \text{diag}(t_1, \dots, t_n, c(t_n)^{-1}, \dots, c(t_1)^{-1}) \mid t_i \in E^\times\}.$$

Then we have

$$\begin{aligned}\Phi_H &= \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}, \text{ and} \\ \Psi_H &= \{a + r \mid a \in \Phi, r \in \mathbb{Z}\}.\end{aligned}$$

We take the root basis

$$\Delta_H = \{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n\}$$

corresponding to the Borel subgroup  $B_H$  consisting of upper triangular matrices. We let  $C_H$  be the fundamental alcove of  $\mathcal{A}(H, S_H)$ . Then the corresponding affine root basis is

$$\Pi_H = \{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n, -2e_1 + 1\}.$$

For these simple affine roots, the corresponding affine root subgroups are described as follows:

- If  $\alpha = e_i - e_{i+1}$  for  $1 \leq i \leq n-1$ , then

$$U_\alpha = \{I_{2n} + u_{e_i - e_{i+1}}(x) \in H(F) \mid x \in \mathcal{O}_E\},$$

where

$$(u_{e_i - e_{i+1}}(x))_{kl} = \begin{cases} x & ((k, l) = (i, i+1)), \\ c(x) & ((k, l) = (2n-i, 2n-i+1)), \\ 0 & (\text{otherwise}). \end{cases}$$

- If  $\alpha = 2e_n$ , then

$$U_\alpha = \{I_{2n} + u_{e_{2n}}(x) \in H(F) \mid x \in \mathcal{O}_F\},$$

where

$$(u_{e_{2n}}(x))_{kl} = \begin{cases} x & ((k, l) = (n, n+1)), \\ 0 & (\text{otherwise}). \end{cases}$$

- If  $\alpha = -2e_1 + 1$ , then

$$U_\alpha = \{I_{2n} + u_{-2e_1}(x) \in H(F) \mid x \in \mathfrak{p}_F\},$$

where

$$(u_{-2e_1}(x))_{kl} = \begin{cases} x & ((k, l) = (2n, 1)), \\ 0 & (\text{otherwise}). \end{cases}$$

We denote the Iwahori subgroup and its subgroups by  $I_H$ ,  $I_H^+$ , and  $I_H^{++}$ .

For  $y = (y_{ij})_{ij} \in I_H^+$ , we regard its affine simple components  $(y_\alpha)_\alpha \in \bigoplus_{\alpha \in \Pi} V(\dot{\alpha})$  as an element of  $k_E^{\oplus n-1} \oplus k_F \oplus k_F$  by

$$I_H^+/I_H^{++} \cong \bigoplus_{\alpha \in \Pi_H} V(\dot{\alpha}) \cong k_E^{\oplus n-1} \oplus k_F \oplus k_F$$

$$(y_{ij})_{ij} \mapsto (\overline{y_{12}}, \dots, \overline{y_{n-1,n}}, \overline{y_{n,n+1}}, \overline{y_{2n,1}\varpi^{-1}}).$$

We fix a nontrivial additive character  $\psi: k_F \rightarrow \mathbb{C}^\times$ . For  $b \in k_F^\times$ , we define an affine generic character  $\psi'_b: I_H^+ \rightarrow \mathbb{C}^\times$  by

$$\psi'_b(y) := \psi \circ \text{Tr}(\overline{y_{12}} + \dots + \overline{y_{n-1,n}}) \cdot \psi(\overline{y_{n,n+1}} + b\overline{y_{2n,1}\varpi^{-1}}) \text{ for } y = (y_{ij})_{ij} \in I_H^+.$$

Then we have  $N(\psi'_b) = Z_H I_H^+$ . Indeed, by Proposition 2.3, we have

$$\tilde{\Omega} \cong X^*(Z(\widehat{\text{U}_{E/F}(2n)})^{I_F})^{\Sigma_F}.$$

Since  $X^*(Z(\widehat{\text{U}_{E/F}(2n)})^{I_F}) \cong \mathbb{Z}$  and  $\Sigma_F$  acts on it by  $(-1)$ -multiplication, its fixed part is trivial. Hence we have  $N_H(I_H) = I_H$  by Lemma 2.4, and we can check easily that  $N(\psi'_b) \subset N_H(I_H) = I_H$  coincides with  $Z_H I_H^+$ .

For a character  $\omega'$  on  $\text{U}(1)$ , we define the character  $\chi'_{b,\omega'}$  on  $Z_H I_H^+$  by

$$\chi'_{b,\omega'}(z) = \omega'(\overline{z}) \text{ for } z \in Z_H(q), \text{ and}$$

$$\chi'_{b,\omega'}(y) = \psi'_b(y) \text{ for } y \in I_H^+.$$

Let  $\pi'_{b,\omega'}$  be the simple supercuspidal representation of  $\text{U}_{E/F}(2n)(F)$  defined by

$$\pi'_{b,\omega'} := \text{c-Ind}_{Z_H I_H^+}^{H(F)} \chi'_{b,\omega'}.$$

Then, by Proposition 2.7 (3), we can check that the set

$$\{(b, \omega') \mid b \in k_F^\times, \omega' \in \text{U}(1)^*\}$$

parametrizes the set of equivalent classes of simple supercuspidal representations of  $\mathrm{U}_{E/F}(2n)(F)$ .

**2.5. Parametrization: the case of  $\mathrm{U}_{E/F}(2n+1)$ .** In this subsection, we consider the case of

$$\mathrm{U}_{E/F}(2n+1) := \{g \in \mathrm{Res}_{E/F}(\mathrm{GL}_{2n+1}) \mid {}^t c(g)Jg = J\},$$

with

$$J = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \ddots & & \\ & & & \ddots & \\ (-1)^{2n} & & & & \end{pmatrix}.$$

Let  $H := \mathrm{U}_{E/F}(2n+1)$ . Then we have  $\mathrm{U}_{E/F}(2n+1)(F) = \{g \in \mathrm{GL}_{2n+1}(E) \mid {}^t c(g)Jg = J\}$ . We identify the center  $Z_H$  of  $H$  with  $\mathrm{U}_{E/F}(1)$ . We choose  $S_H$  to be the following subgroup of diagonal matrices in  $H$ :

$$S_H(F) = \{t = \mathrm{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) \mid t_i \in F^\times\}.$$

This is a maximal  $F$ -split torus of  $H$ , and its centralizer  $Z_{S_H}$  is given by

$$Z_{S_H}(F) = \{t = \mathrm{diag}(t_1, \dots, t_n, z, c(t_n)^{-1}, \dots, c(t_1)^{-1}) \mid t_i \in E^\times, z \in \mathrm{U}_{E/F}(1)(F)\}.$$

Then we have

$$\Phi_H = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i, \pm 2e_i \mid 1 \leq i \leq n\}, \text{ and}$$

$$\Psi_H = \{a + r \mid a \in \Phi, r \in \mathbb{Z}\}.$$

We take the root basis

$$\Delta_H = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\}$$

corresponding to the Borel subgroup  $B_H$  consisting of upper triangular matrices. We let  $C_H$  be the fundamental alcove of  $\mathcal{A}(H, S_H)$ . Then the corresponding affine root basis is

$$\Pi_H = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n, -2e_1 + 1\}.$$

For these simple affine roots, the corresponding affine root subgroups are described as follows:

- If  $\alpha = e_i - e_{i+1}$  for  $1 \leq i \leq n-1$ , then

$$U_\alpha = \{I_{2n+1} + u_{e_i - e_{i+1}}(x) \in H(F) \mid x \in \mathcal{O}_E\},$$

where

$$(u_{e_i - e_{i+1}}(x))_{kl} = \begin{cases} x & ((k, l) = (i, i+1)), \\ c(x) & ((k, l) = (2n-i+1, 2n-i+2)), \\ 0 & (\text{otherwise}). \end{cases}$$

- If  $\alpha = e_n$ , then

$$U_\alpha = \{I_{2n+1} + u_{e_n}(x, y) \in H(F) \mid x, y \in \mathcal{O}_E, xc(x) = y + c(y)\},$$

where

$$(u_{e_n}(x))_{kl} = \begin{cases} x & ((k, l) = (n, n+1)), \\ c(x) & ((k, l) = (n+1, n+2)), \\ y & ((k, l) = (n, n+2)), \\ 0 & (\text{otherwise}). \end{cases}$$

- If  $\alpha = -2e_1 + 1$ , then

$$U_\alpha = \{I_{2n+1} + u_{-2e_1}(y) \in H(F) \mid y \in \mathfrak{p}_E, y + c(y) = 0\},$$

where

$$(u_{-2e_1}(y))_{kl} = \begin{cases} y & ((k, l) = (2n+1, 1)), \\ 0 & (\text{otherwise}). \end{cases}$$

We denote the Iwahori subgroup and its subgroups by  $I_H$ ,  $I_H^+$ , and  $I_H^{++}$ .

We set

$$k_E^0 := \{x \in k_E \mid \text{Tr}(x) = 0\} = \epsilon^{-1}k_F.$$

For  $y = (y_{ij})_{ij} \in I_H^+$ , we regard its affine simple components  $(y_\alpha)_\alpha \in \bigoplus_{\Pi} V(\dot{\alpha})$  as an element of  $k_E^{\oplus n} \oplus k_E^0$  by

$$\begin{aligned} I_H^+ / I_H^{++} &\cong \bigoplus_{\alpha \in \Pi_H} V(\dot{\alpha}) \cong k_E^{\oplus n} \oplus k_E^0 \\ (y_{ij})_{ij} &\mapsto (\overline{y_{12}}, \dots, \overline{y_{n,n+1}}, \overline{y_{2n+1,1}\varpi^{-1}}). \end{aligned}$$

We fix a nontrivial additive character  $\psi: k_F \rightarrow \mathbb{C}^\times$ . For  $b \in k_F^\times$ , we define an affine generic character  $\psi'_b: I_H^+ \rightarrow \mathbb{C}^\times$  by

$$\psi'_b(y) := \psi \circ \text{Tr}(\overline{y_{12}} + \dots + \overline{y_{n,n+1}}) \cdot \psi(b\epsilon \overline{y_{2n+1,1}\varpi^{-1}}) \text{ for } y = (y_{ij})_{ij} \in I_H^+.$$

Then, as is the even case, we have  $N(\psi'_b) = Z_H I_H^+$ .

For a character  $\omega'$  on  $U(1)$ , we define the character  $\chi'_{b,\omega'}$  on  $Z_H I_H^+$  by

$$\begin{aligned} \chi'_{b,\omega'}(z) &= \omega'(\overline{z}) \text{ for } z \in Z_H(q), \text{ and} \\ \chi'_{b,\omega'}(y) &= \psi'_b(y) \text{ for } y \in I_H^+. \end{aligned}$$

Let  $\pi'_{b,\omega'}$  be the simple supercuspidal representation of  $U_{E/F}(2n+1)(F)$  defined by

$$\pi'_{b,\omega'} := \text{c-Ind}_{Z_H I_H^+}^{H(F)} \chi'_{b,\omega'}.$$

Then, by Proposition 2.7 (3), we can check that the set

$$\{(b, \omega') \mid b \in k_F^\times, \omega' \in U(1)^*\}$$

parametrizes the set of equivalent classes of simple supercuspidal representations of  $U_{E/F}(2n+1)(F)$ .

### 3. CHARACTERS OF SIMPLE SUPERCUSPIDAL REPRESENTATIONS

**3.1. Character, character formula, and their twisted versions.** Let us first recall the characters of representations of a  $p$ -adic reductive group. For a connected reductive group  $G$  over  $F$ , we write  $G^{\text{rs}}(F)$  for the set of regular semisimple elements of  $G(F)$ . This is an open subset of  $G(F)$ . We denote by  $\mathcal{H}(G)$  the set of compactly supported locally constant functions on  $G(F)$ .

**Theorem 3.1** ([HC70]). *Let  $G$  be a connected reductive group over  $F$ . Let  $\pi$  be an irreducible smooth representation of  $G(F)$ . Then there exists a unique locally constant function  $\Theta_\pi$  on  $G^{\text{rs}}(F)$  such that*

$$\text{tr } \pi(f) = \int_{G^{\text{rs}}(F)} \Theta_\pi(g) f(g) dg$$

for every  $f \in \mathcal{H}(G)$  satisfying  $\text{supp}(f) \subset G^{\text{rs}}(F)$ , where  $\text{tr } \pi$  is the distribution character of  $\pi$ .

We call  $\Theta_\pi$  the *character* of  $\pi$ . This function is invariant under conjugation.

When  $\pi$  is a supercuspidal representation which is compactly induced from an open subgroup, we have a following formula to describe its character.

**Theorem 3.2** (Character formula, [Sal88]). *Let  $G$  be a connected reductive group over  $F$  and  $Z$  its center. Let  $K$  be an open subgroup of  $G(F)$  such that  $K$  contains  $Z$  and  $K/Z$  is compact. Let  $\rho$  be a finite-dimensional irreducible smooth representation of  $K$ . We assume that the representation  $\pi := \text{c-Ind}_K^{G(F)} \rho$  is irreducible and supercuspidal. Then, for every  $g \in G^{\text{rs}}(F)$ , we have*

$$\Theta_\pi(g) = \sum_{\substack{y \in K \backslash G(F) \\ ygy^{-1} \in K}} \text{tr}(\rho(ygy^{-1})).$$

Next we consider the twisted situation. For a connected reductive group  $G$  over  $F$  and its automorphism  $\theta$  over  $F$ , we write  $G^{\theta\text{-rs}}(F)$  for the set of  $\theta$ -regular  $\theta$ -semisimple elements of  $G(F)$  (see Sections 3.2 and 3.3 in [KS99] for the definitions of  $\theta$ -semisimplicity and  $\theta$ -regularity). This is an open subset of  $G(F)$ .

**Theorem 3.3** ([Lem10, 5.8 Corollaire]). *Let  $G$  be a connected reductive group over  $F$ . Let  $\theta$  be an automorphism of  $G$  defined over  $F$ . Let  $\pi$  be a  $\theta$ -stable (i.e.,  $\pi \cong \pi^\theta$ ) irreducible smooth representation of  $G(F)$ , and fix an isomorphism  $A: \pi \rightarrow \pi^\theta$ . Then there exists a unique locally constant function  $\Theta_{\pi,\theta}$  on  $G^{\theta\text{-rs}}(F)$  such that*

$$\text{tr } \pi_\theta(f) = \int_{G^{\theta\text{-rs}}(F)} \Theta_{\pi,\theta}(g) f(g) dg$$

for every  $f \in \mathcal{H}(G)$  satisfying  $\text{supp}(f) \subset G^{\theta\text{-rs}}(F)$ , where  $\text{tr } \pi_\theta$  is the  $\theta$ -twisted distribution character of  $\pi$  with respect to  $A$ .

We call  $\Theta_{\pi,\theta}$  the  $\theta$ -twisted character of  $\pi$ . This function is invariant under  $\theta$ -conjugation, and depends on an isomorphism  $A: \pi \cong \pi^\theta$  (determined up to scalar multiple).

Similarly in the standard case, we have a formula for the twisted character of supercuspidal representations.

**Theorem 3.4** (Twisted character formula, [Lem10, 6.2 Théorème]). *Let  $G$  be a reductive group over  $F$  and  $Z$  its center. Let  $\theta$  be an automorphism of  $G$  over  $F$ . Let  $K$  be a  $\theta$ -stable open subgroup of  $G(F)$  such that  $K$  contains  $Z$  and  $K/Z$  is compact. Let  $\rho$  be a finite-dimensional  $\theta$ -stable irreducible smooth representation of  $K$ . We fix an isomorphism  $A: \rho \rightarrow \rho^\theta$ . We assume that the representation  $\pi := \text{c-Ind}_K^{G(F)} \rho$  is irreducible and supercuspidal. Then, for  $g \in G^{\theta\text{-rs}}(F)$ , we have*

$$\Theta_{\pi,\theta}(g) = \sum_{\substack{y \in K \backslash G(F) \\ yg\theta(y)^{-1} \in K}} \text{tr}(\rho(yg\theta(y)^{-1}) \circ A)$$

where  $\Theta_{\pi,\theta}$  is the  $\theta$ -twisted character of  $\pi$  with respect to the isomorphism  $\text{c-Ind}_K^G A: \pi \rightarrow \pi^\theta$ .

**3.2. The case of twisted  $G_{E/F}(2n)$ .** Let  $G$  be  $G_{E/F}(2n)$  and  $K_a$  the subgroup  $ZI^+\langle\varphi_{a^{-1}}\rangle$  of  $G(F)$  for  $a \in k_F^\times$ . Let  $\theta$  be the automorphism of  $G$  over  $F$  defined by

$$\theta(g) = J^t c(g)^{-1} J^{-1}, \text{ where } J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \ddots & & \\ (-1)^{2n-1} & & & \end{pmatrix},$$

and  $c$  is the Galois conjugation of  $E/F$ . For  $g \in G(F)$ , we put  $N(g) := g\theta(g) \in G(F)$ . Let  $\omega$  be a conjugate self-dual character on  $k_E^\times$  (i.e.,  $\omega(z) = \omega(c(z)^{-1})$  for every  $z \in k_E^\times$ ). We note that conjugate self-duality of  $\omega$  is equivalent to that  $\omega$  is trivial on  $\text{Nr}(k_E^\times) = k_F^\times$ .

The automorphism  $\theta$  preserves the subgroups  $I^+$  and  $I^{++}$ , and induces the automorphism of  $I^+/I^{++} \cong k_E^{\oplus 2n}$  defined by

$$(x_1, \dots, x_{2n-1}, x_{2n}) \mapsto (c(x_{2n-1}), \dots, c(x_1), c(x_{2n})).$$

In particular, we have  $\psi_a^\theta = \psi_a$ . On the other hand, by a simple computation, we can check  $\theta(\varphi_{a^{-1}}) = -\varphi_{a^{-1}}^{-1}$ . Therefore

$$\pi_{a,\zeta,\omega}^\theta \cong \text{c-Ind}_{K_a}^{G(F)} \chi_{a,\zeta,\omega}^\theta \cong \text{c-Ind}_{K_a}^{G(F)} \chi_{a,\zeta^{-1},\omega} = \pi_{a,\zeta^{-1},\omega},$$

and  $\pi_{a,\zeta,\omega}$  is conjugate self-dual if and only if  $\zeta = \pm 1$ .

Let  $\zeta \in \{\pm 1\}$ , and we put  $\pi := \pi_{1,\zeta,\omega}$ . We denote  $K_1$ ,  $\chi_{1,\zeta,\omega}$ , and  $\varphi_1$  simply by  $K$ ,  $\chi$ , and  $\varphi$ , respectively. We fix an automorphism  $A := \text{id}$  of  $\chi$ . Then this defines the twisted character  $\Theta_{\pi,\theta}$  of  $\pi$ .

First, we compute the twisted character at  $g \in I^+ \cap G^{\theta\text{-rs}}(F)$  such that  $N(g) = g\theta(g) \in I^+$  is affine generic.

**Lemma 3.5.** *Let  $g \in I^+$  be an element such that  $N(g)$  is affine generic. If  $y \in G(F)$  satisfies  $yg\theta(y)^{-1} \in ZI^+\langle\varphi\rangle$ , then  $y \in ZI\langle\varphi\rangle$ .*

*Proof.* Since  $yg\theta(y)^{-1} \in K$ ,  $N(yg\theta(y)^{-1}) = yN(g)y^{-1}$  belongs to  $K\theta(K) = K$ . Moreover, as  $\det(yN(g)y^{-1}) = \det(N(g))$ ,  $yN(g)y^{-1}$  lies in  $I$ . By the assumption and Lemma 2.5,  $y$  must lie in  $N_G(I) = ZI\langle\varphi\rangle$ .  $\square$

**Lemma 3.6.** *Let  $g \in I^+$  be an element such that  $N(g)$  is affine generic. Then a system of representatives of the set*

$$\{y \in ZI^+\langle\varphi\rangle \setminus ZI\langle\varphi\rangle \mid yg\theta(y)^{-1} \in K\}$$

*is given by*

$$Z'_S(q) := \{\text{diag}(t_1, \dots, t_{2n}) \in Z_S(q) \mid t_1 c(t_{2n}) = \dots = t_{2n} c(t_1), t_n = 1\}.$$

*Proof.* Let  $y := \text{diag}(t_1, \dots, t_{2n}) \in Z_S(q)$  satisfying  $t_n = 1$ . Since

$$\text{val}(\det(yg\theta(y)^{-1})) = 0,$$

we have

$$yg\theta(y)^{-1} \in K \Rightarrow yg\theta(y)^{-1} \in Z(q)I^+.$$

Thus we have  $yg\theta(y)^{-1} \in K$  if and only if the diagonal part of

$$\begin{aligned} yg\theta(y)^{-1} &= \text{diag}(t_1, \dots, t_{2n}) \begin{pmatrix} g_{1,1} & \cdots & g_{1,2n} \\ \vdots & \ddots & \vdots \\ g_{2n,1} & \cdots & g_{2n,2n} \end{pmatrix} \text{diag}(c(t_{2n}), \dots, c(t_1)) \\ &= \begin{pmatrix} t_1 c(t_{2n}) g_{1,1} & t_1 c(t_{2n-1}) g_{1,2} & & \\ & t_2 c(t_{2n-1}) g_{2,2} & \ddots & * \\ & & * & \ddots & t_{2n-1} c(t_1) g_{2n-1,2n} \\ t_{2n} c(t_{2n}) g_{2n,1} & & & & t_{2n} c(t_1) g_{2n,2n} \end{pmatrix} \end{aligned}$$

lies in  $Z(q)(Z_S)_1$ . Therefore the condition  $yg\theta(y)^{-1} \in K$  is equivalent to that  $t_1 c(t_{2n}) = \cdots = t_{2n} c(t_1)$ .  $\square$

**Proposition 3.7.** *Let  $g \in I^+ \cap G^{\theta\text{-rs}}(F)$  be an element such that  $N(g)$  is affine generic. Let  $(g_1, \dots, g_{2n})$  be the affine simple components of  $g$ . Then we have*

$$\Theta_{\pi, \theta}(g) = -\text{KI}_{\text{Nr}(g_1 + c(g_{2n-1})) \cdots \text{Nr}(g_{n-1} + c(g_{n+1})) \text{Tr}(g_n) \text{Tr}(g_{2n})}^{n,0}(\psi; k_E/k_F),$$

where the right-hand side is the Kloosterman sum in Definition A.3.

*Proof.* Note that the affine simple components of  $N(g)$  are given by

$$(g_1 + c(g_{2n-1}), \dots, g_{2n-1} + c(g_1), \text{Tr}(g_{2n})),$$

and the affine genericity of  $N(g)$  means that none of them is zero.

By the twisted character formula (Theorem 3.4) and Lemma 3.6, we can compute the twisted character as follows:

$$\begin{aligned} \Theta_{\pi, \theta}(g) &= \sum_{y \in Z'_S(q)} \chi(yg\theta(y)^{-1}) \\ &= \sum_{\alpha \in k_F^\times} \sum_{\substack{t_1, \dots, t_{2n} \in k_E^\times \\ t_i c(t_{2n+1-i}) = \alpha \\ t_n = 1}} \psi \circ \text{Tr} \left( \frac{t_1 c(t_{2n-1}) g_1}{\alpha} + \cdots + \frac{t_{2n-1} c(t_1) g_{2n-1}}{\alpha} + \frac{t_{2n} c(t_{2n}) g_{2n}}{\alpha} \right) \\ &= \sum_{\alpha \in k_F^\times} \sum_{t_1, \dots, t_{n-1} \in k_E^\times} \psi \circ \text{Tr} \left( \frac{t_1}{t_2} g_1 + \cdots + c \left( \frac{t_1}{t_2} \right) g_{2n-1} + \frac{\alpha}{t_1 c(t_1)} g_{2n} \right) \\ &= \sum_{\substack{t_1, \dots, t_{n-1} \in k_E^\times \\ \alpha \in k_F^\times}} \psi \circ \text{Tr} \left( \frac{t_1}{t_2} (g_1 + c(g_{2n-1})) + \cdots + \frac{t_{n-1}}{1} (g_{n-1} + c(g_{n+1})) + \frac{g_n}{\alpha} + \frac{\alpha}{t_1 c(t_1)} g_{2n} \right) \\ &= \text{KI}_{\text{Nr}(g_1 + c(g_{2n-1})) \cdots \text{Nr}(g_{n-1} + c(g_{n+1})) \text{Tr}(g_n) \text{Tr}(g_{2n})}^{n-1,2}(\psi; k_E/k_F) \\ &= -\text{KI}_{\text{Nr}(g_1 + c(g_{2n-1})) \cdots \text{Nr}(g_{n-1} + c(g_{n+1})) \text{Tr}(g_n) \text{Tr}(g_{2n})}^{n,0}(\psi; k_E/k_F). \end{aligned}$$

Here, we used Corollary A.6 in the last equality.  $\square$

Next, we compute the twisted character  $\Theta_{\pi, \theta}$  at  $\varphi_u g$ , where  $g \in I^+$  and  $u \in k_F^\times$  such that  $-N(\varphi_u g) = \varphi_u g \varphi_u^{-1} \theta(g) \in I^+$  is affine generic.

**Lemma 3.8.** *Let  $g \in I^+$  be an element such that  $-N(\varphi_u g) \in I^+$  is affine generic. If  $y \in G(F)$  satisfies  $y\varphi_u g\theta(y)^{-1} \in ZI^+\langle \varphi \rangle$ , then  $y \in ZI\langle \varphi \rangle$ .*



*Proof.* Since  $y\varphi_u g\theta(y)^{-1} \in K$ ,  $N(y\varphi_u g\theta(y)^{-1}) = yN(\varphi_u g)y^{-1}$  belongs to  $K\theta(K) = K$ . Moreover, as  $\det(yN(\varphi_u g)y^{-1}) = \det(N(\varphi_u g))$ ,  $yN(\varphi_u g)y^{-1}$  lies in  $I$ . By the assumption and Lemma 2.5,  $y$  must lie in  $ZI\langle\varphi\rangle$ .  $\square$

**Lemma 3.9.** *Let  $g \in I^+$  be an element such that  $-N(\varphi_u g) \in I^+$  is affine generic. Then a system of representatives of the set*

$$\{y \in ZI^+\langle\varphi\rangle \setminus ZI\langle\varphi\rangle \mid y\varphi_u g\theta(y)^{-1} \in K\}$$

*is given by*

$$Z_S''(q) := \{\text{diag}(t_1, \dots, t_{2n}) \in Z_S(q) \mid t_1 c(t_{2n-1}) = \dots = t_{2n-1} c(t_1) = u, t_{2n} = 1\}.$$

*Proof.* Let  $y = \text{diag}(t_1, \dots, t_{2n}) \in Z_S(q)$  satisfying  $t_{2n} = 1$ . Since

$$\text{val}(\det(y\varphi_u g\theta(y)^{-1})) = \text{val}(\det(\varphi)),$$

we have

$$y\varphi_u g\theta(y)^{-1} \in K = ZI^+\langle\varphi\rangle \Rightarrow \varphi^{-1}y\varphi_u g\theta(y)^{-1} \in Z(q)I^+.$$

Thus we have  $y\varphi_u g\theta(y)^{-1} \in K$  if and only if the diagonal part of

$$\begin{aligned} & \varphi^{-1}y\varphi_u \cdot g \cdot \theta(y)^{-1} \\ &= \text{diag}(t_{2n}u, t_1, \dots, t_{2n-1}) \begin{pmatrix} g_{1,1} & \dots & g_{1,2n} \\ \vdots & \ddots & \vdots \\ g_{2n,1} & \dots & g_{2n,2n} \end{pmatrix} \text{diag}(c(t_{2n}), \dots, c(t_1)) \\ &= \begin{pmatrix} t_{2n}c(t_{2n})ug_{1,1} & t_{2n}c(t_{2n-1})ug_{1,2} & & \\ & t_1c(t_{2n-1})g_{2,2} & \ddots & * \\ & & \ddots & t_{2n-2}c(t_1)g_{2n-1,2n} \\ t_{2n-1}c(t_{2n})g_{2n,1} & * & & t_{2n-1}c(t_1)g_{2n,2n} \end{pmatrix} \end{aligned}$$

lies in  $Z(q)(Z_S)_1$ . Therefore that  $y\varphi_u g\theta(y)^{-1} \in K$  is equivalent to that  $t_1 c(t_{2n-1}) = \dots = t_{2n-1} c(t_1) = u$ .  $\square$

**Proposition 3.10.** *Let  $g \in I^+$  be an element such that  $\varphi_u g \in G^{\theta\text{-rs}}(F)$  and  $-N(\varphi_u g)$  is affine generic. Let  $(g_1, \dots, g_{2n})$  be the affine simple components of  $g$ . Then we have*

$$\Theta_{\pi, \theta}(\varphi_u g) = \zeta \cdot \text{Kl}_{\text{Nr}(ug_1 + c(g_{2n})) \text{Nr}(g_2 + c(g_{2n-1})) \dots \text{Nr}(g_n + c(g_{n+1})) / u}^{n, 0}(\psi; k_E/k_F).$$

*Proof.* Note that the affine simple components of  $-N(\varphi_u g)$  are given by

$$(g_2 + c(g_{2n-1}), g_3 + c(g_{2n-2}), \dots, g_{2n-1} + c(g_2), u^{-1}g_{2n} + c(g_1), ug_1 + c(g_{2n})),$$

and the affine genericity of  $-N(\varphi_u g)$  means that none of them is zero.

By the twisted character formula (Theorem 3.4) and Lemma 3.9, we can compute the twisted character as follows:

$$\begin{aligned}
\Theta_{\pi,\theta}(\varphi_u g) &= \sum_{y \in Z_S''(q)} \chi(\varphi) \chi(\varphi^{-1} y \varphi_u g \theta(y)^{-1}) \\
&= \zeta \sum_{\substack{t_1, \dots, t_{2n} \in k_E^\times \\ t_i c(t_{2n-i}) = u \\ t_{2n} = 1}} \psi \circ \text{Tr} \left( \frac{t_{2n} c(t_{2n-1}) u g_1}{u} + \dots + \frac{t_{2n-2} c(t_1) g_{2n-1}}{u} + \frac{t_{2n-1} c(t_{2n}) g_{2n}}{u} \right) \\
&= \zeta \sum_{\substack{t_1, \dots, t_n \in k_E^\times \\ \text{Nr}(t_n) = u}} \psi \circ \text{Tr} \left( \frac{u}{t_1} g_1 + \dots + c\left(\frac{t_1}{t_2}\right) g_{2n-1} + \frac{g_{2n}}{c(t_1)} \right) \\
&= \zeta \sum_{\substack{t_1, \dots, t_n \in k_E^\times \\ \text{Nr}(t_n) = u}} \psi \circ \text{Tr} \left( \frac{1}{t_1} (u g_1 + c(g_{2n})) + \frac{t_1}{t_2} (g_2 + c(g_{2n-1})) + \dots + \frac{t_{n-1}}{t_n} (g_n + c(g_{n+1})) \right) \\
&= \zeta \cdot \text{KI}_{\text{Nr}(u g_1 + c(g_{2n})) \text{Nr}(g_2 + c(g_{2n-1})) \dots \text{Nr}(g_n + c(g_{n+1}))/u}^{n,0}(\psi; k_E/k_F).
\end{aligned}$$

□

**3.3. The case of twisted  $G_{E/F}(2n+1)$ .** Let  $G$  be  $G_{E/F}(2n+1)$  and  $K_{a\epsilon}$  the subgroup  $ZI^+ \langle \varphi_{a^{-1}\epsilon^{-1}} \rangle$  of  $G(F)$  for  $a \in k_F^\times$ . Let  $\theta$  be the automorphism of  $G$  over  $F$  defined by

$$\theta(g) = J^t c(g)^{-1} J^{-1}, \text{ where } J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & & \ddots & \\ (-1)^{2n} & & & \end{pmatrix},$$

and  $c$  is the Galois conjugation of  $E/F$ . For  $g \in G(F)$ , we put  $N(g) := g\theta(g) \in G(F)$ . Let  $\omega$  be a conjugate self-dual character on  $k_E^\times$ .

The automorphism  $\theta$  preserves the subgroups  $I^+$  and  $I^{++}$ , and induces the automorphism of  $I^+/I^{++} \cong k_E^{\oplus 2n+1}$  defined by

$$(x_1, \dots, x_{2n}, x_{2n+1}) \mapsto (c(x_{2n}), \dots, c(x_1), -c(x_{2n+1})).$$

In particular, we have  $\psi_{a\epsilon}^\theta = \psi_{a\epsilon}$ . On the other hand, by a simple computation, we can check  $\theta(\varphi_{a^{-1}\epsilon^{-1}}) = -\varphi_{a^{-1}\epsilon^{-1}}^{-1}$ . Therefore

$$\pi_{a\epsilon, \zeta, \omega}^\theta \cong \text{c-Ind}_{K_{a\epsilon}}^{G(F)} \chi_{a\epsilon, \zeta, \omega}^\theta \cong \text{c-Ind}_{K_{a\epsilon}}^{G(F)} \chi_{a\epsilon, \zeta^{-1}, \omega} = \pi_{a\epsilon, \zeta^{-1}, \omega},$$

and  $\pi_{a\epsilon, \zeta, \omega}$  is conjugate self-dual if and only if  $\zeta = \pm 1$ .

Let  $\zeta \in \{\pm 1\}$ , and we put  $\pi := \pi_{\epsilon, \zeta, \omega}$ . We denote  $K_\epsilon$  and  $\chi_{\epsilon, \zeta, \omega}$  simply by  $K$  and  $\chi$  respectively. We fix an automorphism  $A := \text{id}$  of  $\chi$ . Then this defines the twisted character  $\Theta_{\pi, \theta}$  of  $\pi$ .

First, we compute the twisted character at  $g \in I^+ \cap G^{\theta\text{-rs}}(F)$  such that  $N(g) = g\theta(g) \in I^+$  is affine generic.

The following lemma follows from the same argument in the proof of Lemma 3.5.

**Lemma 3.11.** *Let  $g \in I^+$  be an element such that  $N(g)$  is affine generic. If  $y \in G(F)$  satisfies  $yg\theta(y)^{-1} \in ZI^+ \langle \varphi_{\epsilon^{-1}} \rangle$ , then  $y \in ZI \langle \varphi_{\epsilon^{-1}} \rangle$ .*

**Lemma 3.12.** *Let  $g \in I^+$  be an element such that  $N(g)$  is affine generic. Then a system of representatives of the set*

$$\{y \in ZI^+ \langle \varphi_{\epsilon^{-1}} \rangle \setminus ZI \langle \varphi_{\epsilon^{-1}} \rangle \mid yg\theta(y)^{-1} \in K\}$$

*is given by*

$$Z'_S(q) := \{\text{diag}(t_1, \dots, t_{2n+1}) \in Z_S(q) \mid t_1 c(t_{2n+1}) = \dots = t_{2n+1} c(t_1), t_{n+1} = 1\}.$$

*Proof.* Let  $y := \text{diag}(t_1, \dots, t_{2n+1}) \in Z_S(q)$  satisfying  $t_{n+1} = 1$ . Since

$$\text{val}(\det(yg\theta(y)^{-1})) = 0,$$

we have

$$yg\theta(y)^{-1} \in K \Rightarrow yg\theta(y)^{-1} \in Z(q)I^+.$$

Thus we have  $yg\theta(y)^{-1} \in K$  if and only if the diagonal part of

$$\begin{aligned} & yg\theta(y)^{-1} \\ &= \text{diag}(t_1, \dots, t_{2n+1}) \begin{pmatrix} g_{1,1} & \cdots & g_{1,2n+1} \\ \vdots & \ddots & \vdots \\ g_{2n+1,1} & \cdots & g_{2n+1,2n+1} \end{pmatrix} \text{diag}(c(t_{2n+1}), \dots, c(t_1)) \\ &= \begin{pmatrix} t_1 c(t_{2n+1}) g_{1,1} & t_1 c(t_{2n}) g_{1,2} & & \\ & t_2 c(t_{2n}) g_{2,2} & \ddots & * \\ & & \ddots & t_{2n} c(t_1) g_{2n,2n+1} \\ t_{2n+1} c(t_{2n+1}) g_{2n+1,1} & * & & t_{2n+1} c(t_1) g_{2n+1,2n+1} \end{pmatrix} \end{aligned}$$

lies in  $Z(q)(Z_S)_1$ . Therefore the condition  $yg\theta(y)^{-1} \in K$  is equivalent to that  $t_1 c(t_{2n+1}) = \dots = t_{2n+1} c(t_1)$ .  $\square$

**Proposition 3.13.** *Let  $g \in I^+ \cap G^{\theta\text{-rs}}(F)$  be an element such that  $N(g)$  is affine generic. Let  $(g_1, \dots, g_{2n+1})$  be the affine simple components of  $g$ . Then we have*

$$\Theta_{\pi, \theta}(g) = K_{\text{Nr}(g_1 + c(g_{2n})) \cdots \text{Nr}(g_n + c(g_{n+1})) \text{Tr}(\epsilon_{g_{2n+1}})}^{n,1}(\psi; k_E/k_F).$$

*Proof.* Note that the affine simple components of  $N(g)$  are given by

$$(g_1 + c(g_{2n}), \dots, g_{2n} + c(g_1), g_{2n+1} - c(g_{2n+1})),$$

and the affine genericity of  $N(g)$  means that none of them is zero.

By the twisted character formula (Theorem 3.4) and Lemma 3.12, we can compute the twisted character as follows:

$$\begin{aligned}
\Theta_{\pi, \theta}(g) &= \sum_{y \in Z'_S(q)} \chi(yg\theta(y)^{-1}) \\
&= \sum_{\substack{t_1, \dots, t_{2n+1} \in k_E^\times \\ t_i c(t_{2n+2-i}) = 1 \\ t_{n+1} = 1}} \psi \circ \text{Tr}(t_1 c(t_{2n})g_1 + \dots + t_{2n} c(t_1)g_{2n} + \epsilon t_{2n+1} c(t_{2n+1})g_{2n+1}) \\
&= \sum_{t_1, \dots, t_n \in k_E^\times} \psi \circ \text{Tr}\left(\frac{t_1}{t_2}g_1 + \dots + c\left(\frac{t_1}{t_2}\right)g_{2n} + \frac{\epsilon}{t_1 c(t_1)}g_{2n+1}\right) \\
&= \sum_{t_1, \dots, t_n \in k_E^\times} \psi \circ \text{Tr}\left(\frac{t_1}{t_2}(g_1 + c(g_{2n})) + \dots + \frac{t_n}{1}(g_n + c(g_{n+1})) + \frac{\epsilon}{t_1 c(t_1)}g_{2n+1}\right) \\
&= \text{Kl}_{\text{Nr}(g_1 + c(g_{2n})) \dots \text{Nr}(g_n + c(g_{n+1})) \text{Tr}(\epsilon g_{2n+1})}^{n,1}(\psi; k_E/k_F).
\end{aligned}$$

□

Next, we compute the twisted character  $\Theta_{\pi, \theta}$  at  $\varphi_{\epsilon^{-1}u}g$ , where  $g \in I^+$  and  $u \in k_F^\times$  such that  $-N(\varphi_{\epsilon^{-1}u}g) = \varphi_{\epsilon^{-1}u}g\varphi_{\epsilon^{-1}u}^{-1}\theta(g) \in I^+$  is affine generic.

The following lemma follows from the same argument in the proof of Lemma 3.8.

**Lemma 3.14.** *Let  $g \in I^+$  be an element such that  $-N(\varphi_{\epsilon^{-1}u}g) \in I^+$  is affine generic. If  $y \in G(F)$  satisfies  $y\varphi_{\epsilon^{-1}u}g\theta(y)^{-1} \in ZI^+\langle\varphi_{\epsilon^{-1}}\rangle$ , then  $y \in ZI\langle\varphi_{\epsilon^{-1}}\rangle$ .*

**Lemma 3.15.** *Let  $g \in I^+$  be an element such that  $-N(\varphi_{\epsilon^{-1}u}g) \in I^+$  is affine generic. Then a system of representatives of the set*

$$\{y \in ZI^+\langle\varphi_{\epsilon^{-1}}\rangle \setminus ZI\langle\varphi_{\epsilon^{-1}}\rangle \mid y\varphi_{\epsilon^{-1}u}g\theta(y)^{-1} \in K\}$$

is given by

$$Z''_S(q) := \{\text{diag}(t_1, \dots, t_{2n+1}) \in Z_S(q) \mid t_1 c(t_{2n}) = \dots = t_{2n} c(t_1) = u, t_{2n+1} = 1\}.$$

*Proof.* Let  $y = \text{diag}(t_1, \dots, t_{2n+1}) \in Z_S(q)$  satisfying  $t_{2n+1} = 1$ . Since

$$\text{val}(\det(y\varphi_{\epsilon^{-1}u}g\theta(y)^{-1})) = \text{val}(\det(\varphi_{\epsilon^{-1}})),$$

we have

$$y\varphi_{\epsilon^{-1}u}g\theta(y)^{-1} \in K = ZI^+\langle\varphi_{\epsilon^{-1}}\rangle \Rightarrow \varphi_{\epsilon^{-1}}^{-1}y\varphi_{\epsilon^{-1}u}g\theta(y)^{-1} \in Z(q)I^+.$$

Thus we have  $y\varphi_{\epsilon^{-1}u}g\theta(y)^{-1} \in K$  if and only if the diagonal part of

$$\begin{aligned}
&\varphi_{\epsilon^{-1}}^{-1}y\varphi_{\epsilon^{-1}u} \cdot g \cdot \theta(y)^{-1} \\
&= \text{diag}(t_{2n+1}u, t_1, \dots, t_{2n}) \begin{pmatrix} g_{1,1} & \dots & g_{1,2n+1} \\ \vdots & \ddots & \vdots \\ g_{2n+1,1} & \dots & g_{2n+1,2n+1} \end{pmatrix} \text{diag}(c(t_{2n+1}), \dots, c(t_1)) \\
&= \begin{pmatrix} t_{2n+1}c(t_{2n+1})ug_{1,1} & t_{2n+1}c(t_{2n})ug_{1,2} & & \\ & t_1c(t_{2n})g_{2,2} & \ddots & * \\ & & \ddots & t_{2n-1}c(t_1)g_{2n,2n+1} \\ t_{2n}c(t_{2n+1})g_{2n+1,1} & * & & t_{2n}c(t_1)g_{2n+1,2n+1} \end{pmatrix}
\end{aligned}$$

lies in  $Z(q)(Z_S)_1$ . Therefore that  $y\varphi_{\epsilon^{-1}u}g\theta(y)^{-1} \in K$  is equivalent to that  $t_1c(t_{2n}) = \dots = t_{2n}c(t_1) = u$ .  $\square$

**Proposition 3.16.** *Let  $g \in I^+$  be an element such that  $\varphi_{\epsilon^{-1}u}g \in G^{\theta\text{-rs}}(F)$  and  $-N(\varphi_{\epsilon^{-1}u}g)$  is affine generic. Let  $(g_1, \dots, g_{2n+1})$  be the affine simple components of  $g$ . Then we have*

$$\Theta_{\pi, \theta}(\varphi_{\epsilon^{-1}u}g) = \zeta \cdot \text{Kl}_{\text{Nr}(ug_1 - \epsilon c(g_{2n+1})) \text{Nr}(g_2 + c(g_{2n})) \dots \text{Nr}(g_n + c(g_{n+2})) \text{Tr}(g_{n+1})/u}^{n,1}(\psi; k_E/k_F).$$

*Proof.* Note that the affine simple components of  $-N(\varphi_{\epsilon^{-1}u}g)$  are given by

$$(g_2 + c(g_{2n}), g_3 + c(g_{2n-1}), \dots, g_{2n} + c(g_2), \epsilon u^{-1}g_{2n+1} + c(g_1), \epsilon^{-1}ug_1 - c(g_{2n+1})),$$

and the affine genericity of  $N(g)$  means that none of them is zero.

By the twisted character formula (Theorem 3.4) and Lemma 3.15, we can compute the twisted character as follows:

$$\begin{aligned} \Theta_{\pi, \theta}(\varphi_{\epsilon^{-1}u}g) &= \sum_{y \in Z_S''(g)} \chi(\varphi_{\epsilon^{-1}}) \chi(\varphi_{\epsilon^{-1}}^{-1} y \varphi_{\epsilon^{-1}u} g \theta(y)^{-1}) \\ &= \zeta \sum_{\substack{t_1, \dots, t_{2n+1} \in k_E^\times \\ t_i c(t_{2n+1-i}) = u \\ t_{2n+1} = 1}} \psi \circ \text{Tr} \left( \frac{t_{2n+1}c(t_{2n})ug_1}{u} + \dots + \frac{t_{2n-1}c(t_1)g_{2n}}{u} + \epsilon \frac{t_{2n}c(t_{2n+1})g_{2n+1}}{u} \right) \\ &= \zeta \sum_{t_1, \dots, t_n \in k_E^\times} \psi \circ \text{Tr} \left( \frac{u}{t_1} g_1 + \dots + c \left( \frac{t_1}{t_2} \right) g_{2n} + \epsilon \frac{g_{2n+1}}{c(t_1)} \right) \\ &= \zeta \sum_{t_1, \dots, t_n \in k_E^\times} \psi \circ \text{Tr} \left( \frac{1}{t_1} (ug_1 - \epsilon c(g_{2n+1})) + \frac{t_1}{t_2} (g_2 + c(g_{2n})) + \dots + \frac{t_n c(t_n)}{u} g_{n+1} \right) \\ &= \zeta \cdot \text{Kl}_{\text{Nr}(ug_1 - \epsilon c(g_{2n+1})) \text{Nr}(g_2 + c(g_{2n})) \dots \text{Nr}(g_n + c(g_{n+2})) \text{Tr}(g_{n+1})/u}^{n,1}(\psi; k_E/k_F). \end{aligned}$$

$\square$

**3.4. The case of  $\text{U}_{E/F}(2n)$ .** Let  $H$  be  $\text{U}_{E/F}(2n)$ . We take  $b \in k_F^\times$  and a character  $\omega'$  on  $\text{U}(1)$ . Let  $K'$  be the subgroup  $Z_H I_H^+$  of  $H(F)$ . We put  $\pi' := \pi'_{b, \omega'}$ . We denote  $\chi'_{b, \omega'}$  simply by  $\chi'$ .

We compute the character of  $\pi'$  at an affine generic element  $h \in I_H^+ \cap H^{\text{rs}}(F)$ .

**Lemma 3.17.** *Let  $h \in I_H^+$  be an affine generic element. If  $y \in H(F)$  satisfies  $yhy^{-1} \in \text{U}_{E/F}(1)I_H^+$ , then  $y \in I_H$ .*

*Proof.* If  $yhy^{-1} \in \text{U}_{E/F}(1)I_H^+ \subset I_H$ , then  $y \in N_H(I_H) = I_H$  by Lemma 2.5.  $\square$

**Proposition 3.18.** *Let  $h \in I_H^+ \cap H^{\text{rs}}(F)$  be an affine generic element with its affine simple components  $(h_1, \dots, h_{n-1}, h_n, h_{2n})$ . Then we have*

$$\Theta_{\pi'}(h) = -\text{Kl}_{\text{Nr}(h_1) \dots \text{Nr}(h_{n-1})h_n h_{2n}b}^{n,0}(\psi; k_E/k_F).$$

*Proof.* By the character formula (Theorem 3.2) and Lemma 3.17, we can compute the character as follows:

$$\begin{aligned}
\Theta_{\pi'}(h) &= \sum_{y \in \mathrm{U}(1)I_H^+ \setminus I_H} \chi'(yhy^{-1}) = \sum_{t \in \mathrm{U}(1) \setminus Z_{S_H}(q)} \chi'(tht^{-1}) \\
&= \sum_{\substack{t_1, \dots, t_{n-1} \in k_E^\times \\ t_n \in k_E^\times / \mathrm{U}(1)}} \psi \circ \mathrm{Tr} \left( \frac{t_1}{t_2} h_1 + \dots + \frac{t_{n-1}}{t_n} h_{n-1} \right) \cdot \psi \left( t_n c(t_n) h_n + \frac{b}{t_1 c(t_1)} h_{2n} \right) \\
&= \sum_{\substack{t_1, \dots, t_{n-1} \in k_E^\times \\ t_n \in k_E^\times / \mathrm{U}(1)}} \psi \circ \mathrm{Tr} \left( \frac{t_1}{t_2} h_1 + \dots + \frac{t_{n-1}}{1} h_{n-1} \right) \cdot \psi \left( t_n c(t_n) h_n + \frac{b}{t_1 c(t_1) t_n c(t_n)} h_{2n} \right) \\
&= \sum_{\substack{t_1, \dots, t_{n-1} \in k_E^\times \\ s_n \in k_F^\times}} \psi \circ \mathrm{Tr} \left( \frac{t_1}{t_2} h_1 + \dots + \frac{t_{n-1}}{1} h_{n-1} \right) \cdot \psi \left( s_n h_n + \frac{b}{t_1 c(t_1) s_n} h_{2n} \right) \\
&= \mathrm{Kl}_{\mathrm{Nr}(h_1) \dots \mathrm{Nr}(h_{n-1}) h_n h_{2n} b}^{n-1, 2}(\psi; k_E/k_F) \\
&= -\mathrm{Kl}_{\mathrm{Nr}(h_1) \dots \mathrm{Nr}(h_{n-1}) h_n h_{2n} b}^{n, 0}(\psi; k_E/k_F).
\end{aligned}$$

Here, we replaced  $t_i$  with  $t_i t_n$  in the 4th equality, and used Corollary A.6 in the last equality.  $\square$

**3.5. The case of  $\mathrm{U}_{E/F}(2n+1)$ .** Let  $H$  be  $\mathrm{U}_{E/F}(2n+1)$ . We take  $b \in k_F^\times$  and a character  $\omega'$  on  $\mathrm{U}(1)$ . Let  $K'$  be the subgroup  $Z_H I_H^+$  of  $H(F)$ . We put  $\pi' := \pi'_{b, \omega'}$ . We denote  $\chi'_{b, \omega'}$  simply by  $\chi'$ .

We compute the character of  $\pi'$  at an affine generic element  $h \in I_H^+ \cap H^{\mathrm{rs}}(F)$ .

By the same argument in the proof of Lemma 3.17, we can show the following lemma.

**Lemma 3.19.** *Let  $h \in I_H^+$  be an affine generic element. If  $y \in H(F)$  satisfies  $yhy^{-1} \in \mathrm{U}_{E/F}(1)I_H^+$ , then  $y \in I_H$ .*

**Proposition 3.20.** *Let  $h \in I_H^+ \cap H^{\mathrm{rs}}(F)$  be an affine generic element with its affine simple components  $(h_1, \dots, h_n, h_{2n+1})$ . Then we have*

$$\Theta_{\pi'}(h) = \mathrm{Kl}_{\mathrm{Nr}(h_1) \dots \mathrm{Nr}(h_n) h_{2n+1} b\epsilon}^{n, 1}(\psi; k_E/k_F).$$

*Proof.* By the character formula (Theorem 3.2) and Lemma 3.19, we can compute the character as follows:

$$\begin{aligned}
\Theta_{\pi'}(h) &= \sum_{y \in \mathrm{U}(1)I_H^+ \setminus I_H} \chi'(yhy^{-1}) = \sum_{t \in \mathrm{U}(1) \setminus Z_{S_H}(q)} \chi'(tht^{-1}) \\
&= \sum_{t_1, \dots, t_n \in k_E^\times} \psi \circ \mathrm{Tr} \left( \frac{t_1}{t_2} h_1 + \dots + \frac{t_n}{1} h_n \right) \cdot \psi \left( \frac{b\epsilon}{t_1 c(t_1)} h_{2n} \right) \\
&= \mathrm{Kl}_{\mathrm{Nr}(h_1) \dots \mathrm{Nr}(h_n) h_{2n+1} b\epsilon}^{n, 1}(\psi; k_E/k_F).
\end{aligned}$$

$\square$

#### 4. TWISTED ENDOSCOPY FOR $G_{E/F}(N)$

**4.1. Simple endoscopic data.** We set  $G = G_{E/F}(N) = \text{Res}_{E/F} \text{GL}_N$ , and define the automorphism  $\theta$  of  $G$  over  $F$  as follows:

$$\theta(g) = J^t c(g)^{-1} J^{-1}, J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \ddots & & \\ (-1)^{N-1} & & & \end{pmatrix}.$$

Then we have

$$\widehat{G} = \text{GL}_N(\mathbb{C}) \times \text{GL}_N(\mathbb{C}),$$

and the Weil group  $W_F$  and the dual automorphism  $\hat{\theta}$  to  $\theta$  act on  $\widehat{G}$  by the following way:

- For  $w \in W_F$  and  $(g, h) \in \text{GL}_N(\mathbb{C}) \times \text{GL}_N(\mathbb{C})$ ,

$$w(g, h) = \begin{cases} (g, h) & (w \in W_E), \\ (h, g) & (w \in W_F \setminus W_E). \end{cases}$$

- For  $(g, h) \in \text{GL}_N(\mathbb{C}) \times \text{GL}_N(\mathbb{C})$ ,

$$\hat{\theta}(g, h) = (J^t h^{-1} J^{-1}, J^t g^{-1} J^{-1}).$$

We set  $H = \text{U}_{E/F}(N)$ . Then we have

$$\widehat{H} = \text{GL}_N(\mathbb{C}),$$

and the Weil group  $W_F$  acts on  $\widehat{H}$  by the following way:

$$\text{For } w \in W_F \text{ and } g \in \text{GL}_N(\mathbb{C}), w(g) = \begin{cases} g & (w \in W_E), \\ J^t g^{-1} J^{-1} & (w \in W_F \setminus W_E). \end{cases}$$

In this paper, we consider the following endoscopic data  $(H, {}^L H, s, \xi)$  for  $(G, \theta, 1)$ .

- (1) The standard base change embedding: We define an  $L$ -embedding

$$\xi_{+1}: {}^L \text{U}_{E/F}(N) \hookrightarrow {}^L G_{E/F}(N)$$

as follows:

$$\begin{aligned} g \rtimes 1 &\mapsto (g, J^t g^{-1} J^{-1}) \rtimes 1, \\ 1 \rtimes w &\mapsto (I_N, I_N) \rtimes w, \text{ for } w \in W_F. \end{aligned}$$

Then  $(H, {}^L H, s = 1, \xi_{+1})$  are endoscopic data for the triplet  $(G, \theta, 1)$ .

- (2) The twisted base change embedding: We fix  $w_c \in W_F \setminus W_E$ . We define the character  $\chi$  on  $E^\times$  by

$$E^\times \ni x \mapsto \chi(x) = (-1)^{\text{val}(x)}.$$

This character is unramified, and  $\chi|_{F^\times}$  is quadratic. We regard  $\chi$  also as a character of  $W_E$  via the local class field theory. Then we define an  $L$ -embedding

$$\xi_{-1}: {}^L \text{U}_{E/F}(N) \hookrightarrow {}^L G_{E/F}(N)$$

as follows:

$$\begin{aligned} g \rtimes 1 &\mapsto (g, J^t g^{-1} J^{-1}) \rtimes 1, \\ 1 \rtimes \sigma &\mapsto (\chi(\sigma) I_N, \chi(\sigma)^{-1} I_N) \rtimes \sigma, \text{ for } \sigma \in W_E, \\ 1 \rtimes w_c &\mapsto (-I_N, I_N) \rtimes w_c. \end{aligned}$$

Then  $(H, {}^L H, s = 1, \xi_{-1})$  are endoscopic data for the triplet  $(G, \theta, 1)$ .

In fact, there are exactly these two kinds of simple endoscopic data for  $G_{E/F}(N)$  up to equivalence (see Section 4.7 in [Rog90]).

**4.2. Norm correspondences.** We recall the norm correspondence for twisted endoscopy in [KS99].

Let  $G$  be a connected quasi-split reductive group over  $F$ , and  $\theta$  an automorphism of  $G$  defined over  $F$ . Let  $(H, {}^L H, s, \xi)$  be endoscopic data for the triplet  $(G, \theta, 1)$ . Then we have the map

$$\mathcal{A}_{H/G}: Cl_{ss}(H) \rightarrow Cl_{\theta-ss}(G, \theta)$$

from the set of semisimple conjugacy classes in  $H(\overline{F})$  to the set of  $\theta$ -semisimple  $\theta$ -conjugacy classes in  $G(\overline{F})$  (see Section 3.3 in [KS99]).

Let  $G^{\theta-srs}(F)$  be the set of strongly  $\theta$ -regular  $\theta$ -semisimple elements in  $G(F)$ , and  $H^{srs}(F)$  the set of strongly regular semisimple elements in  $H(F)$ . We say that  $y \in H^{srs}(F)$  is a *norm* of  $x \in G^{\theta-srs}(F)$  if  $x$  corresponds to  $y$  via the map  $\mathcal{A}_{H/G}$ .

We consider this correspondence for  $G = G_{E/F}(N)$ ,  $H = U_{E/F}(N)$ , and  $\xi = \xi_\kappa$  in the previous section (note that the definition of the map  $\mathcal{A}_{H/G}$  is independent of  $\kappa \in \{\pm 1\}$ ). For  $g \in G(\overline{F})$ , we put  $N(g) := g\theta(g) \in G(\overline{F})$ .

We fix the isomorphisms as follows:

$$\begin{aligned} G_{E/F}(N) \otimes_F \overline{F} &\cong GL_N(\overline{F}) \times GL_N(\overline{F}) \\ g \otimes a &\mapsto (ga, c(g)a), \text{ and} \\ U_{E/F}(N) \otimes_F \overline{F} &\cong GL_N(\overline{F}) \\ h \otimes b &\mapsto hb. \end{aligned}$$

Let  $T_0 := Z_S$  in Section 2.3, and  $T_{H,0} := Z_{S_H}$  in Sections 2.4 and 2.5. These groups are identified with subgroups of diagonal matrices in  $GL_N(\overline{F}) \times GL_N(\overline{F})$  and  $GL_N(\overline{F})$  under the above isomorphisms, respectively. Then we can write the map  $\mathcal{A}_{H/G}$  explicitly with  $T_0$  and  $T_{H,0}$  as follows:

$$\begin{aligned} Cl_{ss}(H) &\xleftarrow{\cong} T_{H,0}(\overline{F})/\Omega_{T_{H,0}} \xrightarrow{\cong} (T_0)_\theta(\overline{F})/\Omega_{T_0}^\theta \xrightarrow{\cong} Cl_{\theta-ss}(G, \theta) \xleftarrow{\cong} T_0(\overline{F}) \\ \text{diag}\left(\frac{t_1}{s_N}, \dots, \frac{t_N}{s_1}\right) &\longleftrightarrow (\text{diag}(t_1, \dots, t_N), \text{diag}(s_1, \dots, s_N)), \end{aligned}$$

where  $\Omega_{T_{H,0}}$  is the Weyl group of  $T_{H,0}$  in  $H$ ,  $\Omega_{T_0}^\theta$  is the  $\theta$ -fixed part of the Weyl group of  $T_0$  in  $G$ , and  $(T_0)_\theta$  is the  $\theta$ -coinvariant part of  $T_0$ . Note that the map  $\mathcal{A}_{H/G}$  is an isomorphism since  $T_{H,0}(\overline{F}) \cong (T_0)_\theta(\overline{F})$  and  $\Omega_{T_{H,0}} \cong \Omega_{T_0}^\theta$  in our situation.

Let us prove some lemmas needed later.

**Lemma 4.1.** *Let  $g \in I^+ \subset G(F) = GL_N(E)$  be an affine generic element. Then  $g$  is regular semisimple elliptic.*

*Proof.* The characteristic polynomial of  $g - I_N$  is Eisenstein, hence it is irreducible over  $E$ . Therefore  $g$  is regular semisimple elliptic.  $\square$



**Lemma 4.2.** *Let  $\gamma \in H^{\text{sts}}(F)$  and  $\delta \in G^{\theta\text{-sts}}(F)$  such that  $\gamma$  is a norm of  $\delta$ . Then, for  $z \in E^\times$ , we have  $z/c(z) \cdot \gamma \in H^{\text{sts}}(F)$  and  $z\delta \in G^{\theta\text{-sts}}(F)$ . Moreover,  $z/c(z) \cdot \gamma$  is a norm of  $z\delta$ .*

*Proof.* From the definition of strongly regular semisimplity and strongly  $\theta$ -regular  $\theta$ -semisimplity, we get the first assertion. The second assertion follows from the above explicit description of the map  $\mathcal{A}_{H,G}$ .  $\square$

**Lemma 4.3.** *Let  $g \in G(\overline{F})$  be a semisimple element such that  $N(g) = g\theta(g)$  is regular semisimple. If  $g\theta(g) = \theta(g)g$ , then  $g$  is  $\theta$ -semisimple.*

*Proof.* Since  $F$  has characteristic zero,  $g$  is  $\theta$ -semisimple if and only if  $\text{Int}(g) \circ \theta$  preserves a pair  $(B_G, T_G)$  in  $G$  (see Section 1.1 in [KS99]). If there exists  $x \in G(\overline{F})$  such that  $xg\theta(x)^{-1} \in T_0(\overline{F})$ , then the pair  $(x^{-1}B_0x, x^{-1}T_0x)$  is  $\text{Int}(g) \circ \theta$ -invariant. Therefore it suffices to prove the existence of such  $x \in G(\overline{F})$ .

As  $g$  is semisimple,  $\theta(g)$  is also semisimple. Since  $g\theta(g) = \theta(g)g$ , there exists  $x \in G(\overline{F})$  such that  $xgx^{-1} \in T_0(\overline{F})$  and  $x\theta(g)x^{-1} \in T_0(\overline{F})$ . Moreover we have  $xN(g)x^{-1} \in T_0(\overline{F})$ . Since the element  $N(g)$  is  $\theta$ -invariant, we can replace  $x$  so that  $xN(g)x^{-1} \in T_0^\theta(\overline{F})$  by  $\Omega_{T_0}$ -conjugation.

Then we have

$$xN(g)x^{-1} = \theta(xN(g)x^{-1}) = \theta(x)N(g)\theta(x)^{-1},$$

hence  $x^{-1}\theta(x)$  belongs to  $\text{Cent}_G(N(g))$ . By the regularity of  $N(g)$ ,  $\text{Cent}_G(N(g)) = x^{-1}T_0x$ . Therefore  $x\theta(x)^{-1} \in T_0(\overline{F})$ , and  $xg\theta(x)^{-1} = xgx^{-1} \cdot x\theta(x)^{-1} \in T_0(\overline{F})$ . This element  $x$  is a desired one.  $\square$

**Lemma 4.4.** *Let  $x$  be a  $\theta$ -semisimple element in  $G(\overline{F})$  and  $y$  a semisimple element in  $H(\overline{F})$  which corresponds to  $x$  via  $\mathcal{A}_{H/G}$ . If  $N(x) = x\theta(x) \in G(\overline{F})$  is regular, then  $x$  is strongly  $\theta$ -regular and  $y$  is strongly regular.*

*Proof.* Since  $\text{Cent}_G(x, \theta) \subset \text{Cent}_G(N(x))$ ,  $x$  is a strongly  $\theta$ -regular (recall the definition of strongly  $\theta$ -regular elements). By the assumption that  $y$  corresponds to  $x$ ,  $y$  is also strongly regular ([KS99, Lemma 3.3.C.]).  $\square$

**Lemma 4.5.** *Let  $h$  be a strongly regular semisimple elliptic element in  $H(F)$ . Then there exists a strongly  $\theta$ -regular  $\theta$ -semisimple  $\theta$ -elliptic element  $g \in G(F)$  such that  $h$  is a norm of  $g$ .*

*Proof.* This follows from the adjoint relation of the transfer factor for strongly regular semisimple elliptic elements (see the proof of [Mok15, Proposition 3.1.1] and [Art13, Proposition 2.1.1]).  $\square$

We now prove the following proposition on the norm correspondence for affine generic elements.

**Proposition 4.6.** *Let  $h \in I_H^+ \subset H(F)$  be an affine generic element. Then  $h$  is strongly regular semisimple elliptic, and there exists  $g \in G(F)$  satisfying the following conditions:*

- $g$  is strongly  $\theta$ -regular  $\theta$ -semisimple  $\theta$ -elliptic, and
- $h = N(g)$ , in particular  $h$  is a norm of  $g$ .

*Proof.* If we regard  $h$  as an element of  $I^+ \subset G(F)$ , then it is an affine generic element of  $G$ . Therefore, by Lemma 4.1, it follows that  $h$  is semisimple. Since the centralizer of  $h$  in  $H(F)$  is compact by Lemma 3.17,  $h$  is elliptic.

We take a  $\theta$ -semisimple element  $x \in G(\overline{F})$  corresponding to  $h$  via  $\mathcal{A}_{H/G}$ , then  $N(x)$  and  $h$  are conjugate in  $H(\overline{F})$ . By  $\theta$ -conjugation, we can replace  $x$  with  $x' \in G(\overline{F})$  satisfying  $N(x') = h$ . By Lemma 4.1,  $N(x') = h$  is a strongly regular element of  $G(F)$ . Therefore  $h$  is a strongly regular element of  $H(F)$  by Lemma 4.4.

Finally, we take a strongly  $\theta$ -regular  $\theta$ -semisimple  $\theta$ -elliptic element  $g' \in G(F)$  such that  $h$  is a norm of  $g'$ . Such an element exists by Lemma 4.5. Then  $h$  and  $N(g')$  are conjugate in  $G(\overline{F})$ , therefore also in  $G(F)$ . Hence we can replace  $g'$  with  $g \in G(F)$  satisfying  $N(g) = h'$  by  $\theta$ -conjugation. This element is a desired one.  $\square$

**4.3. Transfer factors.** In this subsection, we consider  $G = \mathrm{G}_{E/F}(N)$ ,  $H = \mathrm{U}_{E/F}(N)$ , and the standard base change embedding  $\xi = \xi_{+1}$ .

We fix the following  $\theta$ -stable Whittaker datum  $(B_0, \lambda)$  of  $G$ :

- $B_0$  is the subgroup of upper triangular matrices in  $G$ , and
- $\lambda$  is the character of the unipotent radical  $N_0(F)$  of  $B_0(F)$  defined by

$$\lambda(x) = \psi_F \circ \mathrm{Tr}_{E/F}(x_{12} + \cdots + x_{N-1,N}) \text{ for } x = (x_{ij}) \in N_0(F),$$

where  $\psi_F$  is a fixed nontrivial additive character of  $F$ .

Then we have the normalized absolute transfer factor  $\Delta_{H,G}$  for  $G$  and  $H$ . This is a function

$$\Delta_{H,G}: H^{\mathrm{srs}}(F) \times G^{\theta\text{-srs}}(F) \rightarrow \mathbb{C},$$

which has the following properties.

- The value  $\Delta_{H,G}(\gamma, \delta)$  is nonzero only if  $\gamma$  is a norm of  $\delta$ .
- If  $\gamma_1, \gamma_2 \in H^{\mathrm{srs}}(F)$  are stably conjugate, then  $\Delta_{H,G}(\gamma_1, \delta) = \Delta_{H,G}(\gamma_2, \delta)$ .
- If  $\delta_1, \delta_2 \in G^{\theta\text{-srs}}(F)$  are  $\theta$ -conjugate, then  $\Delta_{H,G}(\gamma, \delta_1) = \Delta_{H,G}(\gamma, \delta_2)$ .

Our purpose in this subsection is to compute  $\Delta_{H,G}$ . The normalized transfer factor  $\Delta_{H,G}$  is defined as the product of  $\Delta_{\mathrm{I}}$ ,  $\Delta_{\mathrm{II}}$ ,  $\Delta_{\mathrm{III}}$ ,  $\Delta_{\mathrm{IV}}$  and a constant  $\varepsilon$  (see Section 5.3 in [KS99] for the definition of  $\varepsilon$ ). To define these four factors, we have to fix several auxiliary data. We explain a part of them in general setting (see Sections 4 and 5 in [KS99] for the details). Note that  $s$  is in fact trivial in our situation.

Let  $\hat{\theta}$  be an automorphism of  $\widehat{G}$  which is dual to  $\theta$ . We first fix

- a  $(\Gamma, \hat{\theta})$ -stable splitting  $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$  of  $\widehat{G}$  and
- a  $\Gamma$ -stable splitting  $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_H\})$  of  $\widehat{H}$

such that  $s \in \mathcal{T}$  (this condition is trivial in our setting) and  $\xi(\mathcal{T}_H) = (\mathcal{T}^{\hat{\theta}})^0$ .

We take  $\delta \in G^{\theta\text{-srs}}(F)$ , and let  $\gamma \in H^{\mathrm{srs}}(F)$  be a norm of  $\delta$ . For these elements, we fix

- a  $\theta$ -stable pair  $(B, T)$  of  $G$  and
- a pair  $(B_H, T_H)$  of  $H$ , where  $T_H := \mathrm{Cent}_H(\gamma)$

such that the isomorphism

$$T_H \cong T_{\theta}$$

which is induced by pairs  $(B, T)$ ,  $(\mathcal{B}, \mathcal{T})$ ,  $(B_H, T_H)$ , and  $(\mathcal{B}_H, \mathcal{T}_H)$  is defined over  $F$ . This isomorphism is called an admissible embedding, and we can take such pairs (see [KS99, Lemma 3.3.B]). Then, by the definition of the norm correspondence, there exist  $g \in G_{\mathrm{sc}}(\overline{F})$  and  $\delta^* \in T(\overline{F})$  satisfying the following conditions:

- $\gamma$  has the image  $\delta^*$  under  $T_H \cong T_\theta$  and
- $\delta^* = g\delta\theta(g)^{-1}$ .

We write  $R(G, T)$  for the set of roots of  $T$  in  $G$ . Let  $R_{\text{res}}$  denote the set of restricted roots of  $R(G, T)$ :

$$R_{\text{res}} := \{\alpha_{\text{res}} := \alpha|_{(T^\theta)^0} \mid \alpha \in R(G, T)\},$$

and  $R_{\text{res}}^\vee$  those of  $R(\widehat{G}, \mathcal{T})$ . We fix an  $a$ -data and a  $\chi$ -data for  $R_{\text{res}}$ . By the bijection  $\alpha_{\text{res}} \mapsto (\alpha^\vee)_{\text{res}}$  from  $R_{\text{res}}$  to  $R_{\text{res}}^\vee$ , we transport the  $\Gamma$ -action,  $a$ -data, and  $\chi$ -data of  $R_{\text{res}}$  to  $R_{\text{res}}^\vee$ . Here, we identify  $R_{\text{res}}^\vee$  with the set of restricted coroots of  $T$  in  $G$ :

$$\{(\alpha^\vee)_{\text{res}} := \alpha^\vee|_{(\widehat{T}^\theta)_0} \mid \alpha^\vee \in R^\vee(G, T)\},$$

by the canonical isomorphism  $R^\vee(G, T) \cong R(\widehat{G}, \mathcal{T})$ .

**Lemma 4.7.** *Let  $\gamma \in H^{\text{srs}}(F)$  and  $\delta \in G^{\theta\text{-srs}}(F)$ . If  $\gamma$  is a norm of  $\delta$ , then we have*

$$\Delta_{\text{I}}(\gamma, \delta) = 1.$$

*Proof.* Let  $\langle, \rangle$  be the Tate-Nakayama pairing between  $H^1(F, T_{\text{sc}}^\theta)$  and  $\pi_0(\widehat{T_{\text{sc}}^\theta}^\Gamma)$ . Let  $s_{T, \theta}$  be the image of  $s$  under the projection

$$\mathcal{T} \cong \widehat{T} \twoheadrightarrow (\widehat{T}_{\text{ad}})_\theta \cong \widehat{T_{\text{sc}}^\theta}.$$

In fact  $s_{T, \theta}$  is invariant under  $\Gamma$  ([KS99, Lemma 4.2]). We denote the image of  $s_{T, \theta}$  under the map

$$\widehat{T_{\text{sc}}^\theta}^\Gamma \twoheadrightarrow \pi_0(\widehat{T_{\text{sc}}^\theta}^\Gamma)$$

by  $\mathbf{s}$ . Then  $\Delta_{\text{I}}(\gamma, \delta)$  is defined to be  $\langle \lambda, \mathbf{s} \rangle$ , where  $\lambda$  is a 1-cocycle determined by the fixed Whittaker data and  $a$ -data (see Section 4.2 in [KS99]). Since  $s = 1$  in our setting,  $\Delta_{\text{I}}(\gamma, \delta)$  is trivial.  $\square$

**Lemma 4.8.** *Let  $\gamma \in H^{\text{srs}}(F)$  and  $\delta \in G^{\theta\text{-srs}}(F)$ . If  $\gamma$  is a norm of  $\delta$ , then we have*

$$\Delta_{\text{II}}(\gamma, \delta) = 1.$$

*Proof.* The set of restricted roots  $R_{\text{res}}$  is reduced. Moreover the set of restricted coroots  $R_{\text{res}}^\vee$  coincides with the set  $R^\vee(H, T_H)$  of coroots of  $T_H$  in  $H$ . Therefore  $\Delta_{\text{II}}(\gamma, \delta)$  is trivial by Lemma 4.3.A in [KS99].  $\square$

We next recall the definition of  $\Delta_{\text{III}}$ . Let  $\langle, \rangle$  be the Tate-Nakayama pairing

$$H^1(F, T_{\text{sc}} \rightarrow T) \times H^1(W_F, \widehat{T} \rightarrow \widehat{T}_{\text{ad}}) \rightarrow \mathbb{C},$$

where the maps  $T_{\text{sc}} \rightarrow T$  and  $\widehat{T} \rightarrow \widehat{T}_{\text{ad}}$  are  $1 - \theta$  and  $1 - \hat{\theta}$ , respectively. Then the third factor  $\Delta_{\text{III}}(\gamma, \delta)$  is defined by  $\langle \text{inv}(\gamma, \delta), \mathbf{A}_0 \rangle$ , where  $\text{inv}(\gamma, \delta)$  and  $\mathbf{A}_0$  are 1-hypercocycles defined as follows:

- Let  $v_0: \Gamma \rightarrow T_{\text{sc}}(\overline{F})$  be a map defined by  $v_0(\sigma) = g\sigma(g)^{-1}$ . In fact this is a 1-cocycle and defines a 1-hypercocycle

$$(v_0^{-1}, \delta^*) \in Z^1(F, T_{\text{sc}} \rightarrow T).$$

We denote by  $\text{inv}(\gamma, \delta)$  the image of  $(v_0^{-1}, \delta^*)$  in  $H^1(F, T_{\text{sc}} \rightarrow T)$ .

- Let  $\widehat{G}^1 := (\widehat{G}^\theta)^0$ ,  ${}^L G^1 := \widehat{G}^1 \rtimes W_F$ , and  $\mathcal{T}^1 := \mathcal{T} \cap \widehat{G}^1$ . Since  $R(\widehat{G}^1, \mathcal{T}^1) \subset R_{\text{res}}^\vee$ , the  $\Gamma$ -action and  $\chi$ -data for  $R(\widehat{G}^1, \mathcal{T}^1)$  are induced from those of  $R_{\text{res}}^\vee$ . These data define an  $L$ -embedding  ${}^L(T_\theta) \hookrightarrow {}^L G^1$  (see Section 2.6 in [LS87]). By composing the fixed admissible embedding  ${}^L(T_H) \cong {}^L(T_\theta)$  and a natural embedding  ${}^L G^1 \hookrightarrow {}^L G$ , we get

$$\xi_1: {}^L(T_H) \cong {}^L(T_\theta) \hookrightarrow {}^L G^1 \hookrightarrow {}^L G.$$

On the other hand, since  $s \in \mathcal{T}$ , we also have  $\mathcal{T}^1 \subset \xi(\widehat{H})$  and  $R(\xi(\widehat{H}), \mathcal{T}^1) \subset R_{\text{res}}^\vee$ . Similarly as above, we get an  $L$ -embedding  ${}^L(T_H) \hookrightarrow {}^L H$ . By composing  $\xi$ , we get

$$\xi_2: {}^L(T_H) \hookrightarrow {}^L H \xrightarrow{\xi} {}^L G.$$

Then these two  $L$ -embeddings define a 1-cocycle  $a_T: W_F \rightarrow \mathcal{T}$  such that  $\xi_2 = \xi_1 \cdot a_T$ . We regard  $a_T$  as a 1-cocycle to  $\widehat{T}$  via  $\mathcal{T} \cong \widehat{T}$ . In fact this 1-cocycle defines a 1-hypercocycle

$$(a_T^{-1}, s) \in Z^1(W_F, \widehat{T} \rightarrow \widehat{T}_{\text{ad}}).$$

We denote by  $\mathbf{A}_0$  the image of  $(a_T^{-1}, s)$  in  $H^1(W_F, \widehat{T} \rightarrow \widehat{T}_{\text{ad}})$ .

**Lemma 4.9.** *Let  $\gamma \in H^{\text{srs}}(F)$  and  $\delta \in G^{\theta\text{-srs}}(F)$ . If  $\gamma$  is a norm of  $\delta$ , then we have*

$$\Delta_{\text{III}}(\gamma, \delta) = 1.$$

*Proof.* In our setting,  $s$  is trivial. Hence  $\xi(H)$  is exactly equal to  $\widehat{G}^1$ , and we have  $a_T(\sigma) = \xi(1 \rtimes \sigma)$  for every  $\sigma \in \Gamma$ . As we took  $\xi = \xi_{+1}$ , the 1-cocycle  $a_T$  is trivial. Hence so is  $\mathbf{A}_0$ , and we have  $\Delta_{\text{III}}(\gamma, \delta) = 1$ .  $\square$

Finally, we consider the fourth factor. Let  $D_H$  be the Weyl discriminant:

$$D_H(\gamma) := |\det(\text{Ad}(\gamma) - 1 | \mathfrak{h}/\mathfrak{t}_H)|^{\frac{1}{2}} \text{ for } \gamma \in H^{\text{srs}}(F),$$

where  $\mathfrak{h}$  and  $\mathfrak{t}_H$  are the Lie algebras of  $H$  and  $T_H$ , respectively. Let  $D_{G,\theta}$  be the twisted Weyl discriminant:

$$D_{G,\theta}(\delta) := |\det(\text{Ad}(\delta) \circ \theta - 1 | \mathfrak{g}/\mathfrak{t})|^{\frac{1}{2}} \text{ for } \delta \in G^{\theta\text{-srs}}(F),$$

where  $\mathfrak{g}$  and  $\mathfrak{t}$  are the Lie algebras of  $G$  and  $\text{Cent}_G(\text{Cent}_G(\delta, \theta)^0)$ , respectively. The fourth factor is defined by

$$\Delta_{\text{IV}}(\gamma, \delta) = D_{G,\theta}(\delta)/D_H(\gamma).$$

**Lemma 4.10.** *Let  $\gamma \in H^{\text{srs}}(F)$  and  $\delta \in G^{\theta\text{-srs}}(F)$ . If  $\gamma$  is a norm of  $\delta$ , then we have*

$$\Delta_{\text{IV}}(\gamma, \delta) = 1$$

(note that this is equivalent to  $D_{G,\theta}(\delta) = D_H(\gamma)$ ).

*Proof.* By [KS99, Lemma 4.5.A] and the same argument in the proof of Lemma 4.8, we get the result.  $\square$

**Proposition 4.11.** *Let  $\gamma \in H^{\text{srs}}(F)$ ,  $\delta \in G^{\theta\text{-srs}}(F)$ . If  $\gamma$  is a norm of  $\delta$ , then the transfer factor  $\Delta_{H,G}(\gamma, \delta)$  is equal to  $\varepsilon$ .*

*Proof.* Since

$$\Delta_{H,G}(\gamma, \delta) = \Delta_{\text{I}}(\gamma, \delta) \Delta_{\text{II}}(\gamma, \delta) \Delta_{\text{III}}(\gamma, \delta) \Delta_{\text{IV}}(\gamma, \delta) \varepsilon,$$

we get the result by Lemmas 4.7, 4.8, 4.9 and 4.10.  $\square$

*Remark 4.12.* We can compute the transfer factors also for the case of twisted base change embedding  $\xi = \xi_{-1}$ . In this case, we have

$$\Delta_{H,G}(\gamma, \delta) = \varepsilon \cdot \chi((\det(\delta)) = \varepsilon \cdot (-1)^{\text{val}(\det(\delta))}.$$

We note that we can also deduce this equality from our main theorem (Theorems 5.13 and 6.13) and the endoscopic character relation.

**4.4. Conjugate self-dual  $L$ -parameters.** Let  $G$  and  $H$  be as in Section 4.1. We denote by  $\Phi(\text{GL}_N(E))$  and  $\Phi(G)$  the set of equivalence classes of  $L$ -parameters of  $\text{GL}_N$  over  $E$  and  $G$ , respectively. We fix  $w_c \in W_F \setminus W_E$  and define a map  $\phi \mapsto \phi'$  from  $\Phi(\text{GL}_N(E))$  to  $\Phi(G)$  as:

$$\begin{aligned} \phi' : L_F = W_F \times \text{SL}_2(\mathbb{C}) &\rightarrow (\text{GL}_N(\mathbb{C}) \times \text{GL}_N(\mathbb{C})) \rtimes W_F = {}^L G \\ \sigma &\mapsto (\phi(\sigma), \phi(w_c^{-1} \sigma w_c)) \rtimes \sigma \text{ for } \sigma \in L_E, \\ w_c &\mapsto (\phi(w_c^2), I_N) \rtimes w_c. \end{aligned}$$

Then this map is bijective and independent of the choice of  $w_c$  (see Section 4.7 in [Rog90]). We often identify  $\Phi(\text{GL}_N(E))$  with  $\Phi(G)$  via this map.

**Definition 4.13.** We say that an  $L$ -parameter  $\phi \in \Phi(G)$  is conjugate self-dual if  $\phi^\vee$  is equivalent to  $\phi^c$  as representations of  $L_E$ , where

$$\begin{aligned} \phi^\vee(\sigma) &:= {}^t \phi(\sigma)^{-1}, \text{ and} \\ \phi^c(\sigma) &:= \phi(w_c^{-1} \sigma w_c). \end{aligned}$$

If an  $L$ -parameter  $\phi \in \Phi(G)$  is conjugate self-dual, then there exists  $A \in \text{GL}_N(\mathbb{C})$  such that

$$\begin{aligned} {}^t \phi^c(\sigma) \cdot A \cdot \phi(\sigma) &= A \text{ for } \sigma \in L_E, \text{ and} \\ {}^t A &= \eta \cdot \phi(w_c^2) \cdot A, \end{aligned}$$

where  $\eta = \pm 1$ . In fact this  $\eta$  depends only on  $\phi$ , and we call  $\eta$  the *parity* of  $\phi$ .

Next we consider the group  $\text{U}_{E/F}(N)$ . Let  $H := \text{U}_{E/F}(N)$  and  $\xi = \xi_\kappa$  the  $L$ -embedding defined in Section 4.1, where  $\kappa = \pm 1$ . Then  $(H, {}^L H, 1, \xi)$  are endoscopic data for  $(G, \theta, 1)$ .

We define a map  $\xi_{\kappa,*} : \Phi(H) \rightarrow \Phi(G)$  by  $\xi_{\kappa,*}(\phi_H) := \xi_\kappa \circ \phi_H$ . Then the image of this map is described as follows:

- Lemma 4.14** ([Mok15, Lemma 2.2.1]). (1) *The map  $\xi_{\kappa,*}$  is injective and its image is the set of conjugate self-dual  $L$ -parameters of  $G$  with parity  $(-1)^{N-1\kappa}$ .*  
(2) *Let  $\phi_H \in \Phi(H)$  and  $\phi_\kappa := \xi_{\kappa,*}(\phi_H)$ . We denote by  $\pi_{\phi_\kappa}$  the irreducible smooth representation of  $\text{GL}_N(E)$  which corresponds to  $\phi_\kappa$  via the local Langlands correspondence for  $\text{GL}_N(E)$ . Then we have  $\pi_{\phi_{-1}} = \pi_{\phi_{+1}} \otimes (\chi \circ \det)$ .*

*Proof.* The first assertion is Lemma 2.2.1 in [Mok15]. The second assertion follows easily from the definition of the above bijection  $\Phi(G) \cong \Phi(\text{GL}_N(E))$  and a property of the local Langlands correspondence for  $\text{GL}_N$ .  $\square$

**Lemma 4.15.** *Let  $a \in k_E^\times$ ,  $\zeta \in \{\pm 1\}$ , and  $\omega$  a conjugate self-dual character on  $k_E^\times$ . Then we have*

$$\pi_{a,\zeta,\omega} \cong \pi_{a,-\zeta,\omega} \otimes (\chi \circ \det),$$

where  $\chi = (-1)^{\text{val}}$  is the character of  $E^\times$ .

*Proof.* By the Frobenius reciprocity, we have

$$\begin{aligned}
& \text{Hom}_{G(F)}(\pi_{a,\zeta,\omega}, \pi_{a,-\zeta,\omega} \otimes (\chi \circ \det)) \\
& \cong \text{Hom}_{ZI^+\langle\varphi_{a-1}\rangle}(\chi_{a,\zeta,\omega}, \pi_{a,-\zeta,\omega} \otimes (\chi \circ \det)|_{ZI^+\langle\varphi_{a-1}\rangle}) \\
& \cong \text{Hom}_{ZI^+\langle\varphi_{a-1}\rangle}(\chi_{a,\zeta,\omega} \otimes (\chi^{-1} \circ \det), \pi_{a,-\zeta,\omega}|_{ZI^+\langle\varphi_{a-1}\rangle}) \\
& = \text{Hom}_{ZI^+\langle\varphi_{a-1}\rangle}(\chi_{a,-\zeta,\omega}, \pi_{a,-\zeta,\omega}|_{ZI^+\langle\varphi_{a-1}\rangle}) \\
& \neq 0.
\end{aligned}$$

Since  $\pi_{a,\zeta,\omega}$  and  $\pi_{a,-\zeta,\omega} \otimes (\chi \circ \det)$  are irreducible, they are equivalent.  $\square$

**4.5. Endoscopic character relation.** We first recall the endoscopic classification of representations of unitary groups in [Mok15].

We write  $\Phi(H)$  for the set of  $L$ -parameters of  $H$ . For  $\phi_H \in \Phi(H)$ , we set

$$\begin{aligned}
S_{\phi_H} &:= \text{Cent}_{\widehat{H}}(\text{Im}(\phi_H)), \text{ and} \\
\mathcal{S}_{\phi_H} &:= S_{\phi_H}/S_{\phi_H}^0 Z(\widehat{H})^\Gamma.
\end{aligned}$$

**Theorem 4.16** ([Mok15, Theorems 2.5.1 and 3.2.1]). *For every bounded  $L$ -parameter  $\phi_H \in \Phi(H)$ , there is a finite set  $\Pi_{\phi_H}$  consisting of irreducible tempered representations of  $H$ , and the following properties hold.*

- There is a bijection from  $\Pi_{\phi_H}$  to the group  $\widehat{\mathcal{S}_{\phi_H}}$  of characters of  $\mathcal{S}_{\phi_H}$ .
- For every  $f \in \mathcal{H}(G)$ , we have the following equality of stable distributions:

$$\text{tr}(\pi_\kappa)_\theta(f) = \sum_{\pi_H \in \Pi_{\phi_H}} \text{tr} \pi_H(f_\kappa^H),$$

where  $\pi_\kappa$  is a representation of  $G(F)$  corresponding to  $\xi_\kappa \circ \phi_H$  via the local Langlands correspondence for  $\text{GL}_N$ ,  $\text{tr}(\pi_\kappa)_\theta$  is its  $\theta$ -twisted distribution character with respect to the normalization determined by the Whittaker datum in Section 4.3, and  $f_\kappa^H \in \mathcal{H}(H)$  is a transfer of  $f$  to  $H$ .

Let  $\pi$  be a conjugate self-dual simple supercuspidal representation of  $G(F)$  such that the parity of the corresponding  $L$ -parameter  $\phi$  is  $(-1)^{N-1}\kappa$ . From Lemma 4.14 (1),  $\phi$  factors through the  $L$ -embedding  $\xi_\kappa$ . We write  $\phi_H$  for the  $L$ -parameter of  $H$  such that  $\phi = \xi_\kappa \circ \phi_H$ . Then, by Theorem 4.16, we get a finite set  $\Pi_{\phi_H}$  consisting of irreducible representations of  $H(F) = \text{U}_{E/F}(F)$ .

Since  $\pi$  is supercuspidal, its  $L$ -parameter  $\phi: W_E \rightarrow \text{GL}_N(\mathbb{C})$  is irreducible as a representation of  $W_E$ . Therefore  $\text{Cent}_{\widehat{G}}(\text{Im}(\phi')) \subset \text{GL}_N(\mathbb{C}) \times \text{GL}_N(\mathbb{C})$  consists of pairs of scalar matrices. Thus  $\text{Cent}_{\widehat{H}}(\text{Im}(\phi_H))$  consists of scalar matrices. Hence the group  $\mathcal{S}_{\phi_H}$  is trivial and  $\Pi_{\phi_H}$  is a singleton by Theorem 4.16.

Moreover, since the  $L$ -packet  $\Pi_{\phi_H}$  contains a supercuspidal representation by [Mœg07, 8.4.4 Théorème], the unique representation in  $\Pi_{\phi_H}$  is supercuspidal. We denote it by  $\pi_H$  and say that  $\pi_H$  is *associated to*  $\pi$ . We remark that the character  $\Theta_{\pi_H}$  of  $\pi_H$  is stable by Theorem 4.16.

Our purpose is to determine  $\pi_H$  by using the relation in Theorem 4.16.

$$\begin{array}{ccc}
\pi & \begin{array}{c} \text{LLC for } \text{GL}_N \\ \curvearrowright \end{array} & \begin{array}{c} W_F \xrightarrow{\phi'} L_G \\ \searrow \phi_H \quad \uparrow \xi_\kappa \\ L_H \end{array} \\
\Pi_{\phi_H} = \{\pi_H\} & \begin{array}{c} \text{Theorem 4.16} \\ \curvearrowright \end{array} & 
\end{array}$$

By using a stable version of the twisted Weyl integration formula, we can rewrite the relation in Theorem 4.16 in terms of the characters  $\Theta_{\pi,\theta}$  and  $\Theta_{\pi_H}$  as follows.

**Theorem 4.17.** *Let  $\pi$  be a conjugate self-dual simple supercuspidal representation of  $G(F)$ . Let  $\pi_H$  be the irreducible representation of  $H(F)$  associated to  $\pi$ . Let  $\Theta_{\pi,\theta}$  be the twisted character of  $\pi$  with respect to the normalizations in Sections 3.2 and 3.3, and  $\Theta_{\pi_H}$  the character of  $\pi_H$ . Let  $c \in \mathbb{C}^\times$  be the ratio of the Whittaker normalization to the normalizations in Sections 3.2 and 3.3. Then, for every  $g \in G^{\theta\text{-srs}}(F)$ , we have the following equality:*

$$c \cdot \Theta_{\pi,\theta}(g) = \sum_{h \rightarrow g} \frac{D_H(h)^2}{D_{G,\theta}(g)^2} \Delta_{H,G}(h, g) \Theta_{\pi_H}(h),$$

where the sum is over stable conjugacy classes of norms  $h \in H^{\text{srs}}(F)$  of  $g$ .

We can simplify this equation by the bijectivity of  $\mathcal{A}_{H/G}$  and the computation of the transfer factor  $\Delta_{H,G}$  in Section 4.3.

**Lemma 4.18.** *Let  $g \in G^{\theta\text{-srs}}(F)$ . Then  $g$  has at most one norm in  $H^{\text{srs}}(F)$  up to stable conjugacy.*

*Proof.* Let  $h, h' \in H^{\text{srs}}(F)$  be norms of  $g$ . In our situation, the norm map  $\mathcal{A}_{H/G}$  is bijective, hence  $h$  and  $h'$  are conjugate in  $H(\overline{F})$ . Since  $h$  and  $h'$  are strongly regular, they are stably conjugate.  $\square$

**Corollary 4.19.** *Let  $\pi$  be a conjugate self-dual simple supercuspidal representation of  $G(F)$ . We suppose that the parity of the  $L$ -parameter  $\phi$  of  $\pi$  is  $(-1)^{N-1}$  (i.e.,  $\phi$  factors through the standard base change  $L$ -embedding  $\xi_{+1}$ ). Let  $g \in G^{\theta\text{-srs}}(F)$  and  $h \in H^{\text{srs}}(F)$  such that  $h$  is a norm of  $g$ . Then we have*

$$c/\varepsilon \cdot \Theta_{\pi,\theta}(g) = \Theta_{\pi_H}(h).$$

*Proof.* Combining Theorem 4.17 with Lemma 4.18, we have

$$c \cdot \Theta_{\pi,\theta}(g) = \frac{D_H(h)^2}{D_{G,\theta}(g)^2} \Delta_{H,G}(h, g) \Theta_{\pi_H}(h).$$

By Proposition 4.11 and Lemma 4.10, we have

$$c/\varepsilon \cdot \Theta_{\pi,\theta}(g) = \Theta_{\pi_H}(h).$$

$\square$

## 5. MAIN THEOREM: THE EVEN CASE

In this section we treat the even case. Let  $G := G_{E/F}(2n)$ , and  $H := U_{E/F}(2n)$ .

5.1. **Norms of  $1 + \varphi_u$  and  $\varphi_u(1 + \varphi_u)$ .** For  $u \in k_F^\times$ , we consider

$$1 + \varphi_u = \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ \varpi u & & & 1 \end{pmatrix} \in I^+ \subset G(F), \text{ and}$$

$$N(1 + \varphi_u) = (1 + \varphi_u)\theta(1 + \varphi_u) = (1 - \varpi u)^{-1} \begin{pmatrix} 1 + \varpi u & 2 & \dots & 2 \\ 2\varpi u & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 2\varpi u & \dots & 2\varpi u & 1 + \varpi u \end{pmatrix}$$

$$\in I_H^+ \subset H(F).$$

These are affine generic elements.

**Proposition 5.1.** *The element  $1 + \varphi_u \in G(F)$  is strongly  $\theta$ -regular  $\theta$ -semisimple and  $N(1 + \varphi_u) \in H(F)$  is strongly regular semisimple. Moreover,  $N(1 + \varphi_u)$  is a norm of  $1 + \varphi_u$ .*

*Proof.* We first show that  $1 + \varphi_u$  is  $\theta$ -semisimple. By Lemma 4.1,  $1 + \varphi_u$  is semisimple. Since

$$\begin{aligned} \theta(1 + \varphi_u) &= J(1 + {}^t\varphi_u)^{-1}J^{-1} \\ &= J(1 - {}^t\varphi_u + {}^t\varphi_u^2 - \dots)J^{-1} \\ &= 1 + \varphi_u + \varphi_u^2 + \dots, \end{aligned}$$

$1 + \varphi_u$  commutes with  $\theta(1 + \varphi_u)$ . Moreover, since the residual characteristic is not equal to 2,  $N(1 + \varphi_u)$  is an affine generic element of  $G(F)$ , hence regular semisimple by Lemma 4.1. Therefore  $1 + \varphi_u$  is  $\theta$ -semisimple by Lemma 4.3.

By the definition of  $\mathcal{A}_{H/G}$ ,  $N(1 + \varphi_u)$  corresponds to  $1 + \varphi_u$  via  $\mathcal{A}_{H/G}$ . Then  $1 + \varphi_u$  is strongly  $\theta$ -regular and  $N(1 + \varphi_u)$  is strongly regular by Lemma 4.4.  $\square$

Next, for  $u \in k_F^\times$  we consider the elements

$$\varphi_u(1 + \varphi_u) \in G(F), \text{ and}$$

$$N(\varphi_u(1 + \varphi_u)) = (\varphi_u(1 + \varphi_u)) \cdot \theta(\varphi_u(1 + \varphi_u)) = -N(1 + \varphi_u) \in H(F).$$

**Proposition 5.2.** *The element  $\varphi_u(1 + \varphi_u) \in G(F)$  is strongly  $\theta$ -regular  $\theta$ -semisimple and  $-N(1 + \varphi_u) \in H(F)$  is strongly regular semisimple. Moreover,  $-N(1 + \varphi_u)$  is a norm of  $\varphi_u(1 + \varphi_u)$ .*

*Proof.* By the same argument in the proof of Proposition 5.1, we only have to show that  $\varphi_u(1 + \varphi_u)$  is  $\theta$ -semisimple. Since the characteristic polynomial of  $\varphi_u$  is  $t^{2n} - \varpi u$  and irreducible over  $E$ ,  $\varphi_u$  is semisimple. Hence so is  $\varphi_u(1 + \varphi_u)$ . Since

$$\begin{aligned} \theta(\varphi_u(1 + \varphi_u)) &= -\varphi_u^{-1}J(1 - {}^t\varphi_u + {}^t\varphi_u^2 - \dots)J^{-1} \\ &= -\varphi_u^{-1}(1 + \varphi_u + \varphi_u^2 + \dots), \end{aligned}$$

$\varphi_u(1 + \varphi_u)$  commutes with  $\theta(\varphi_u(1 + \varphi_u))$ . Finally, as

$$N(\varphi_u(1 + \varphi_u)) = -N(1 + \varphi_u)$$

is regular semisimple by the proof of Proposition 5.1,  $\varphi_u(1 + \varphi_u)$  is  $\theta$ -semisimple by Lemma 4.3.  $\square$



**5.2. Parity of simple supercuspidal representations.** Let  $a \in k_F^\times$ . From Lemmas 4.14 and 4.15, we know that the parity of the  $L$ -parameter  $\pi_{a,1,\omega}$  is different from that of  $\pi_{a,-1,\omega}$ . In this subsection, we determine the parity of these parameters. We first consider the case where  $a = 1$ .

Let  $\zeta_\omega \in \{\pm 1\}$  such that the parity of the  $L$ -parameter of  $\pi_{1,\zeta_\omega,\omega}$  is  $-1$ . We put  $\pi := \pi_{1,\zeta_\omega,\omega}$ . Then the  $L$ -parameter of  $\pi$  factors through the standard base change embedding  $\xi_{+1}$  by Lemma 4.14. Let  $\pi_H$  be the supercuspidal representation of  $H(F)$  which is associated to  $\pi$ .

**Proposition 5.3.** *For  $u \in k_F^\times$  and  $z \in \mathcal{O}_E^\times$ , we have*

$$\Theta_{\pi_H}(z/c(z) \cdot N(1 + \varphi_u)) = c/\varepsilon \cdot \Theta_{\pi,\theta}(z(1 + \varphi_u)) = -c/\varepsilon \cdot \omega(\bar{z}) \cdot \text{Kl}_{2^{2n_u}}^{n,0}(\psi; k_E/k_F).$$

*Proof.* We write  $g$  and  $h$  for  $1 + \varphi_u$  and  $N(1 + \varphi_u)$ , respectively. Then, by Proposition 5.1 and Lemma 4.2,  $z/c(z) \cdot h \in H^{\text{srs}}(F)$  is a norm of  $zg \in G^{\theta\text{-srs}}(F)$ . Therefore we have  $c/\varepsilon \cdot \Theta_{\pi,\theta}(zg) = \Theta_{\pi_H}(z/c(z) \cdot h)$  by Corollary 4.19.

On the other hand, since we have  $\omega_\pi(z) = \omega(\bar{z})$ , where  $\omega_\pi$  is the central character of  $\pi$ ,

$$\Theta_{\pi,\theta}(zg) = \omega(\bar{z}) \cdot \Theta_{\pi,\theta}(g).$$

Since  $N(g)$  is affine generic,  $g$  satisfies the assumption of Proposition 3.7. As the affine simple components of  $g = 1 + \varphi_u$  is  $(1, \dots, 1, u)$ , we have

$$\Theta_{\pi,\theta}(g) = -\text{Kl}_{2^{2n_u}}^{n,0}(\psi; k_E/k_F).$$

□

**Proposition 5.4.** *For  $u \in k_F^\times$ , we have*

$$\Theta_{\pi_H}(-N(1 + \varphi_u)) = c/\varepsilon \cdot \Theta_{\pi,\theta}(\varphi_u(1 + \varphi_u)) = c/\varepsilon \cdot \zeta_\omega \cdot \text{Kl}_{2^{2n_u}}^{n,0}(\psi; k_E/k_F).$$

*Proof.* We write  $g'$  and  $h'$  for  $\varphi_u(1 + \varphi_u)$  and  $-N(1 + \varphi_u)$ , respectively. Then, by Proposition 5.2,  $h' \in H^{\text{srs}}(F)$  is a norm of  $g' \in G^{\theta\text{-srs}}(F)$ . Therefore we have  $c/\varepsilon \cdot \Theta_{\pi,\theta}(g') = \Theta_{\pi_H}(h')$  by Corollary 4.19.

Since  $-N(g')$  is affine generic,  $1 + \varphi_u$  satisfies the assumption of Proposition 3.10. As the affine simple components of  $1 + \varphi_u$  is  $(1, \dots, 1, u)$ , we have

$$\Theta_{\pi,\theta}(\varphi_u(1 + \varphi_u)) = \zeta_\omega \cdot \text{Kl}_{2^{2n_u}}^{n,0}(\psi; k_E/k_F).$$

□

**Corollary 5.5.** *We have  $\zeta_\omega = -\omega(\epsilon)$ .*

*Proof.* We regard  $\epsilon$  as an element of  $\mathcal{O}_E^\times$  by the Teichmüller lift. By Propositions 5.3 and 5.4, we have

$$-\omega(\epsilon) \cdot \text{Kl}_{2^{2n_u}}^{n,0}(\psi; k_E/k_F) = \zeta_\omega \cdot \text{Kl}_{2^{2n_u}}^{n,0}(\psi; k_E/k_F).$$

Since there exists  $u \in k_F^\times$  such that  $\text{Kl}_{2^{2n_u}}^{n,0}(\psi; k_E/k_F) \neq 0$  by Corollary A.5, we get  $\zeta_\omega = -\omega(\epsilon)$ . □

Note that  $\zeta_\omega$  depends only on the character  $\omega$ . Thus the parity of the  $L$ -parameter of  $\pi_{a,\zeta_\omega,\omega}$  is  $-1$  for any  $a \in k_F^\times$ .

**5.3. Existence of  $I_H^{++}$ -fixed vector.** Let  $\pi := \pi_{1, \zeta_\omega, \omega}$  and  $\pi_H$  the supercuspidal representation of  $H(F)$  associated to  $\pi$ .

**Lemma 5.6.** *Let  $h \in H(F)$  be an affine generic element with its affine simple components  $(h_1, \dots, h_n, h_{2n})$ . Then  $\Theta_{\pi_H}(h)$  is equal to either 0 or*

$$-c/\varepsilon \cdot \text{Kl}_{\text{Nr}(h_1) \cdots \text{Nr}(h_{n-1})h_n h_{2n}}^{n,0}(\psi; k_E/k_F).$$

*Proof.* For such  $h$ , we take  $g \in G(F)$  satisfying the conditions in Proposition 4.6. Since  $h \in H^{\text{srs}}(F)$ ,  $g \in G^{\theta\text{-srs}}(F)$ , and  $h$  is a norm of  $g$ , we have

$$c/\varepsilon \cdot \Theta_{\pi, \theta}(g) = \Theta_{\pi_H}(h)$$

by Corollary 4.19. We compute the left-hand side of this equality.

If there is not  $x \in G(F)$  such that  $xg\theta(x)^{-1} \in ZI^+\langle\varphi\rangle$ , then  $\Theta_{\pi, \theta}(g) = 0$  by the twisted character formula (Theorem 3.4).

Let us consider the case where there exists  $x \in G(F)$  such that  $xg\theta(x)^{-1} \in ZI^+\langle\varphi\rangle$ . By Lemma 3.5, we may assume  $x \in Z_S(q)$ . Since  $\theta(\varphi) = -\varphi^{-1}$  and  $\varphi$  normalizes  $I^+$ , we have

$$\varphi^k xg\theta(\varphi^k x)^{-1} \in ZI^+ \amalg ZI^+ \varphi$$

for some  $k \in \mathbb{Z}$ .

We first consider the case where  $\varphi^k xg\theta(\varphi^k x)^{-1} \in ZI^+$ . We take  $z \in Z$  and  $g_0 \in I^+$  such that  $\varphi^k xg\theta(\varphi^k x)^{-1} = zg_0$ . Then we have

$$\begin{aligned} N(\varphi^k xg\theta(\varphi^k x)^{-1}) &= \varphi^k xN(g)(\varphi^k x)^{-1} \\ &= z/c(z) \cdot N(g_0). \end{aligned}$$

Hence we know that  $z/c(z) \in 1 + \mathfrak{p}_E$  by comparing the diagonal parts. Therefore we have  $z \in \text{Nr}_{E/F}(\mathcal{O}_E^\times) \cdot (1 + \mathfrak{p}_E) \cdot \langle\varpi\rangle$ . Since  $\omega$  is conjugate self-dual, we have  $\omega_\pi(z) = 1$ , where  $\omega_\pi$  is the central character of  $\pi = \pi_{1, \zeta_\omega, \omega}$  (note that  $\omega_\pi(\varpi) = 1$  since  $\varpi I_{2n} = \varphi^{2n}$  and  $\chi_{1, \zeta_\omega, \omega}(\varphi) = \pm 1$ ).

Let  $x = \text{diag}(t_1, \dots, t_{2n})$ . Let  $(g_1, \dots, g_{2n})$  be the affine simple components of  $g_0$ . Then the affine simple components of  $N(zg_0) = z/c(z) \cdot N(g_0)$  are

$$(g_1 + c(g_{2n-1}), \dots, g_{2n-1} + c(g_1), g_{2n} + c(g_{2n})).$$

On the other hand, since the affine simple components of  $N(g) = h$  are

$$(h_1, \dots, h_{n-1}, h_n, c(h_{n-1}), \dots, c(h_1), h_{2n}),$$

those of  $N(zg_0) = \varphi^k xN(g)(\varphi^k x)^{-1}$  are some cyclic permutation of

$$\left( \frac{t_1}{t_2} h_1, \dots, \frac{t_{2n-1}}{t_{2n}} c(h_1), \frac{t_{2n}}{t_1} h_{2n} \right).$$

Therefore  $g_0$  satisfies the assumption of Proposition 3.7, and we have

$$\begin{aligned} \Theta_{\pi, \theta}(g) &= \Theta_{\pi, \theta}(\varphi^k xg\theta(\varphi^k x)^{-1}) \\ &= \Theta_{\pi, \theta}(zg_0) \\ &= \omega_\pi(z) \Theta_{\pi, \theta}(g_0) \\ &= -\text{Kl}_{\text{Nr}(g_1 + c(g_{2n-1})) \cdots \text{Nr}(g_{n-1} + c(g_{n+1})) \text{Tr}(g_n) \text{Tr}(g_{2n})}^{n,0}(\psi; k_E/k_F) \\ &= -\text{Kl}_{\text{Nr}(h_1) \cdots \text{Nr}(h_{n-1})h_n h_{2n}}^{n,0}(\psi; k_E/k_F). \end{aligned}$$

We next consider the case where  $\varphi^k x g \theta(\varphi^k x)^{-1} \in Z I^+ \varphi$ . We take  $z \in Z$  and  $g_0 \in I^+$  such that  $\varphi^k x g \theta(\varphi^k x)^{-1} = z \varphi g_0$ . Then we have

$$\begin{aligned} N(\varphi^k x g \theta(\varphi^k x)^{-1}) &= \varphi^k x N(g)(\varphi^k x)^{-1} \\ &= z/c(z) \cdot N(\varphi g_0). \end{aligned}$$

Hence we know that  $z/c(z) \in -(1 + \mathfrak{p}_E)$  by comparing the diagonal parts. Therefore we have  $z \in \epsilon \cdot \text{Nr}_{E/F}(\mathcal{O}_E^\times) \cdot (1 + \mathfrak{p}_E) \cdot \langle \varpi \rangle$  (here, we regard  $\epsilon$  as an element of  $\mathcal{O}_E^\times$  by the Teichmüller lift). Since  $\omega$  is conjugate self-dual, we have  $\omega_\pi(z) = \omega(\epsilon)$ .

Let  $x = \text{diag}(t_1, \dots, t_{2n})$ . Let  $(g_1, \dots, g_{2n})$  be the affine simple components of  $g_0$ . Then the affine simple components of  $N(z \varphi g_0) = z/c(z) \cdot N(\varphi g_0)$  are

$$(g_2 + c(g_{2n-1}), \dots, g_{2n-1} + c(g_2), g_{2n} + c(g_1), g_1 + c(g_{2n})).$$

On the other hand, since the affine simple components of  $N(g) = h$  are

$$(h_1, \dots, h_{n-1}, h_n, c(h_{n-1}), \dots, c(h_1), h_{2n}),$$

those of  $N(z \varphi g_0) = N(\varphi^k x g \theta(\varphi^k x)^{-1}) = \varphi^k x N(g)(\varphi^k x)^{-1}$  are some cyclic permutation of

$$\left( \frac{t_1}{t_2} h_1, \dots, \frac{t_{2n-1}}{t_{2n}} c(h_1), \frac{t_{2n}}{t_1} h_{2n} \right).$$

Therefore  $g_0$  satisfies the assumption of Proposition 3.10, and we have

$$\begin{aligned} \Theta_{\pi, \theta}(g) &= \Theta_{\pi, \theta}(z \varphi g_0) \\ &= \omega_\pi(z) \cdot \Theta_{\pi, \theta}(\varphi g_0) \\ &= \omega(\epsilon) \cdot \zeta_\omega \cdot \text{Kl}_{\text{Nr}(g_1 + c(g_{2n})) \cdots \text{Nr}(g_n + c(g_{n+1}))}^{n, 0}(\psi; k_E/k_F) \\ &= \omega(\epsilon) \cdot \zeta_\omega \cdot \text{Kl}_{\text{Nr}(h_1) \cdots \text{Nr}(h_{n-1}) h_n h_{2n}}^{n, 0}(\psi; k_E/k_F) \\ &= -\text{Kl}_{\text{Nr}(h_1) \cdots \text{Nr}(h_{n-1}) h_n h_{2n}}^{n, 0}(\psi; k_E/k_F). \end{aligned}$$

Here, we used  $\omega_\pi(z) = \omega(\epsilon)$  in the 3rd equality, and Corollary 5.5 in the last equality.  $\square$

**Corollary 5.7.** *The representation  $(\pi_H, V_H)$  has a nonzero  $I_H^{++}$ -fixed vector.*

*Proof.* We take  $u \in k_F^\times$  such that  $\text{Kl}_{2^{2n}u}^{n, 0}(\psi; k_E/k_F) \neq 0$  (such  $u \in k_F^\times$  exists by Corollary A.5). Let  $h := N(1 + \varphi_u) \in I_H^+$  and  $f := \mathbf{1}_{h I_H^{++}}$  the characteristic function of  $h I_H^{++}$ . Since the subgroup  $I_H^{++}$  is normal in  $I_H^+$ , we have  $I_H^{++} h I_H^{++} = h I_H^{++}$ . Moreover  $h I_H^{++}$  is contained in  $H^{\text{rs}}(F)$  by Proposition 4.6. Therefore  $f$  belongs to  $\mathcal{H}(H, I_H^{++})$  and satisfies  $\text{supp}(f) \subset H^{\text{rs}}(F)$ .

By a property of the character (Theorem 3.1), we have

$$\text{tr } \pi_H(f) = \int_{H^{\text{rs}}(F)} f(y) \Theta_{\pi_H}(y) dy = \int_{h I_H^{++}} \Theta_{\pi_H}(y) dy.$$

For  $y \in h I_H^{++}$ ,  $\Theta_{\pi_H}(y)$  is equal to either  $-c/\varepsilon \cdot \text{Kl}_{2^{2n}u}^{n, 0}(\psi; k_E/k_F)$  or 0, by Lemma 5.6. Moreover we have

$$\Theta_{\pi_H}(h) = c/\varepsilon \cdot \Theta_{\pi, \theta}(1 + \varphi_u) = -c/\varepsilon \cdot \text{Kl}_{2^{2n}u}^{n, 0}(\psi; k_E/k_F) \neq 0$$

by Proposition 5.3. Since the character  $\Theta_{\pi_H}$  is locally constant on  $H^{\text{rs}}(F)$ , we have

$$\int_{h I_H^{++}} \Theta_{\pi_H}(y) dy \neq 0.$$

On the other hand, by the definition of the distribution character,

$$\mathrm{tr} \pi_H(f) = \mathrm{tr}(\pi_H(f) | V_H^{I_H^{++}}).$$

Therefore we have  $V_H^{I_H^{++}} \neq 0$ .  $\square$

**5.4. Simple supercuspidality of  $\pi_H$ .** Let  $\pi$  and  $\pi_H$  be as in the previous subsection. In this subsection, we prove that  $\pi_H$  is simple supercuspidal.

By Corollary 5.7, the finite abelian group  $I_H^+/I_H^{++}$  acts on the nonzero subspace  $\pi_H^{I_H^{++}}$  of  $\pi_H$ , and it decomposes into a direct sum of characters of  $I_H^+/I_H^{++}$ . We take a character  $\eta$  of  $I_H^+$  which is contained in it.

**Lemma 5.8.** *If  $\eta$  is an affine generic character of  $I_H^+$ , then  $\pi_H$  is simple supercuspidal.*

*Proof.* By the Frobenius reciprocity, we have

$$\mathrm{Hom}_{H(F)}(\mathrm{c}\text{-Ind}_{I_H^+}^{H(F)} \eta, \pi_H) \cong \mathrm{Hom}_{I_H^+}(\eta, \pi_H|_{I_H^+}) \neq 0.$$

By Proposition 2.7 and Remark 2.6, the representation  $\mathrm{c}\text{-Ind}_{I_H^+}^{H(F)} \eta$  is a finite direct sum of simple supercuspidal representations. Since  $\pi_H$  is irreducible, it is equivalent to one of them.  $\square$

From this lemma, we are reduced to prove the affine genericity of  $\eta$ . We first prove two technical lemmas which will be needed in the proof of the affine genericity of  $\eta$ .

**Lemma 5.9.** *Let  $\beta \in \Pi_H$  be a simple affine root of  $S_H$  in  $H$ . Then the subgroup  $\langle I_H^{++}, U_\beta \rangle$  of  $H(F)$  contains  $H_{\mathbf{x}}^+$  for some point  $\mathbf{x}$  of the apartment  $\mathcal{A}(H, S_H)$ , where  $H_{\mathbf{x}}^+$  is the pro-unipotent radical of the parahoric subgroup of  $H(F)$  associated to  $\mathbf{x}$ .*

*Proof.* We recall that the pro-unipotent radical of the parahoric subgroup associated to a point  $\mathbf{x} \in \mathcal{A}(H, S_H)$  and  $I_H^{++}$  are given by

$$H_{\mathbf{x}}^+ = \langle (Z_{S_H})_1, U_\alpha \mid \alpha \in \Psi_H, \alpha(\mathbf{x}) > 0 \rangle, \text{ and}$$

$$I_H^{++} = \langle (Z_{S_H})_1, U_\alpha \mid \alpha \in \Psi_H^+ \setminus \Pi_H \rangle,$$

respectively. Therefore we have to find a point  $\mathbf{x}$  satisfying the following condition:

(\*) If an affine root  $\alpha$  satisfies  $\alpha(\mathbf{x}) > 0$ , then we have  $\alpha \in (\Psi_H^+ \setminus \Pi_H) \cup \{\beta\}$ .

We first consider the cases where  $\beta = -2e_1 + 1$ . In this case, we take  $\mathbf{x} = 0$ . If  $\alpha = a + r \in \Psi_H$  satisfies  $\alpha(\mathbf{x}) > 0$ , then we have  $r > 0$ . Hence  $\alpha \in \Psi_H^+ \setminus \Pi_H$  or  $\alpha = \beta$ .

We next consider the other cases. We define a point  $\mathbf{x} \in \mathcal{A}(H, S_H)$  as follows:

$$\mathbf{x} := \begin{cases} (\check{e}_1 + \cdots + \check{e}_i)/2 & (\beta = e_i - e_{i+1} \text{ for } 1 \leq i < n), \\ (\check{e}_1 + \cdots + \check{e}_n)/2 & (\beta = 2e_n). \end{cases}$$

It is enough to prove (\*) for  $\alpha = a + r_a \in \Psi_H$ , where  $r_a \in \mathbb{Z}$  is the smallest integer satisfying  $a(\mathbf{x}) + r_a > 0$ .

- If  $a \in \Phi_H^+ \setminus \Delta_H$ , then we have  $0 \leq a(\mathbf{x}) \leq 1$ . Hence  $\alpha(\mathbf{x}) > 0$  implies that  $r_a \geq 0$ . Therefore  $\alpha = a + r_a$  belongs to  $\Psi_H^+ \setminus \Pi_H$ .

- If  $a \in \Delta_H$ , then we have

$$\lceil a(\mathbf{x}) \rceil = \begin{cases} 0 & (a \neq \beta), \\ 1 & (a = \beta). \end{cases}$$

Hence we have

$$\alpha(\mathbf{x}) > 0 \iff \begin{cases} r_a > 0 & (a \neq \beta), \\ r_a \geq 0 & (a = \beta). \end{cases}$$

Therefore we have either  $\alpha \in \Psi_H^+ \setminus \Pi_H$  or  $\alpha = \beta$ .

- If  $a \in \Phi_H^- \setminus \{-2e_1\}$ , then we have  $-1 \leq a(\mathbf{x}) \leq 0$ . Hence  $\alpha(\mathbf{x}) > 0$  implies that  $r_a \geq 1$ . Therefore  $\alpha = a + r_a$  belongs to  $\Psi_H^+ \setminus \Pi_H$ .
- If  $a = -2e_1$ , then we have  $\alpha(\mathbf{x}) = r_a - 1$ . Hence  $\alpha(\mathbf{x}) > 0$  implies that  $r_a \geq 2$ , and we have  $\alpha \in \Psi_H^+ \setminus \Pi_H$ .

□

**Lemma 5.10.** *Let  $\mathbf{x}$  be either 0 or  $(\check{e}_1 + \dots + \check{e}_n)/2$  in the apartment  $\mathcal{A}(H, S_H)$  of  $H(F)$ , and  $H_{\mathbf{x}}$  the hyperspecial subgroup of  $H(F)$  associated to  $\mathbf{x}$ . Let  $y \in H(F)$ . If  $y$  satisfies  $yhy^{-1} \in H_{\mathbf{x}}$  for an affine generic element  $h \in H(F)$ , then  $y \in H_{\mathbf{x}}$ .*

*Proof.* Let  $y \in H(F)$  satisfying  $yhy^{-1} \in H_{\mathbf{x}}$  for an affine generic element  $h \in I_H^+$ . As affine genericity is preserved by  $I_H^+$ -conjugation, any element of  $H_{\mathbf{x}}yI_H^+$  satisfies the same condition as  $y$ . It suffices to show  $H_{\mathbf{x}}yI_H^+ \subset H_{\mathbf{x}}$ .

The isomorphism in Proposition 2.3 induces an isomorphism

$$H_{\mathbf{x}} \backslash H(F) / I_H^+ \cong ((N_{S_H}(F) \cap H_{\mathbf{x}}) / (Z_{S_H})_1) \backslash (N_{S_H}(F) / (Z_{S_H})_1)$$

(see [Ric13, Theorem 1.4]) and the right-hand side is represented by

$$Z_{S_H}(\varpi^{\mathbb{Z}}) := \{ \text{diag}(\varpi^{r_1}, \dots, \varpi^{r_n}, \varpi^{-r_n}, \dots, \varpi^{-r_1}) \mid r_1, \dots, r_n \in \mathbb{Z} \}.$$

Hence we may assume

$$\begin{aligned} y &= \text{diag}(t_1, \dots, t_{2n}) \\ &= \text{diag}(\varpi^{r_1}, \dots, \varpi^{r_n}, \varpi^{-r_n}, \dots, \varpi^{-r_1}) \in Z_{S_H}(\varpi^{\mathbb{Z}}). \end{aligned}$$

Since  $(yhy^{-1})_{ij} = t_i/t_j \cdot h_{ij}$ , and  $\mathbf{x}$  equals either 0 or  $(\check{e}_1 + \dots + \check{e}_n)/2$ , we have the following inequalities:

$$\begin{aligned} r_1 - r_2 &\geq 0, \\ &\vdots \\ r_{n-1} - r_n &\geq 0, \\ 2r_n &\geq -1, \text{ and} \\ -2r_1 &\geq -1. \end{aligned}$$

Therefore  $(r_1, \dots, r_n)$  is  $(0, \dots, 0)$ .

Hence we have  $H_{\mathbf{x}}yI_H^+ = H_{\mathbf{x}}I_H^+ = H_{\mathbf{x}}$ , and this completes the proof. □

**Proposition 5.11.** *Every character  $\eta$  of  $I_H^+$  which is contained in  $\pi_H^{I_H^{++}}$  is affine generic. In particular, the representation  $\pi_H$  is simple supercuspidal.*

*Proof.* We suppose that  $\eta$  is not affine generic.

Since  $\eta$  is not affine generic, there exists a simple affine root  $\beta \in \Pi_H$  such that its affine root subgroup  $U_\beta$  is contained in  $\text{Ker}(\eta)$ . Hence  $\pi_H$  has a nonzero  $\langle I_H^{++}, U_\beta \rangle$ -fixed vector. Then, by Lemma 5.9,  $\pi_H$  also has a nonzero  $H_{\mathbf{y}}^+$ -fixed vector for some point  $\mathbf{y} \in \mathcal{A}(H, S_H)$ , where  $H_{\mathbf{y}}^+$  is the pro-unipotent radical of the parahoric subgroup associated to  $\mathbf{y}$ . Therefore the depth of  $\pi_H$  is zero.

On the other hand,  $\pi_H$  is generic with respect to any Whittaker datum. Indeed, as  $\pi$  is supercuspidal,  $\phi$  is a bounded  $L$ -parameter. Hence  $\phi_H$  is also bounded. Therefore the  $L$ -packet  $\Pi_{\phi_H}$  of  $\phi_H$  contains a generic representation by Corollary 9.2.4 in [Mok15]. However  $\Pi_{\phi_H}$  is a singleton consisting of  $\pi_H$ . Thus  $\pi_H$  is generic.

Then, by Lemma 6.1.2 in [DR09], the generic depth-zero supercuspidal representation  $\pi_H$  can be obtained by the compact induction of a representation of the reduction of the hyperspecial subgroup  $H_{\mathbf{x}}$  of  $H(F)$  for some hyperspecial vertex  $\mathbf{x} \in \mathcal{A}(H, S_H)$ . Since every hyperspecial vertex of the building  $\mathcal{B}(H)$  of  $H$  is conjugate to either 0 or  $(\check{e}_1 + \cdots + \check{e}_n)/2 \in \mathcal{A}(H, S_H)$ , we may assume  $\mathbf{x}$  is one of them (see [Tit79, Section 4]).

Let  $H_{\mathbf{x}}^+$  be the pro-unipotent radical of  $H_{\mathbf{x}}$ . Let  $\pi_H \cong \text{c-Ind}_{H_{\mathbf{x}}^+}^{H(F)} \dot{\rho}$ , where  $\dot{\rho}$  is the inflation of a representation  $\rho$  of  $H_{\mathbf{x}}/H_{\mathbf{x}}^+ \cong \text{U}_{k_E/k_F}(2n)(k_F)$  to  $H_{\mathbf{x}}$ . Then, by the character formula (Theorem 3.2), we have

$$\Theta_{\pi_H}(N(1 + \varphi_u)) = \sum_{\substack{y \in H_{\mathbf{x}} \backslash H(F) \\ yN(1 + \varphi_u)y^{-1} \in H_{\mathbf{x}}}} \text{tr}(\dot{\rho}(yN(1 + \varphi_u)y^{-1}))$$

for any  $u \in k_F^\times$ . By Lemma 5.10, we have

$$\Theta_{\pi_H}(N(1 + \varphi_u)) = \text{tr}(\dot{\rho}(N(1 + \varphi_u))).$$

We first consider the case where  $\mathbf{x} = 0$ . Then the image of  $N(1 + \varphi_u)$  in  $H_{\mathbf{x}}/H_{\mathbf{x}}^+$  is independent of  $u$ . Hence the character  $\Theta_{\pi_H}(N(1 + \varphi_u))$  is constant on  $u \in k_F^\times$ . However we have

$$\Theta_{\pi_H}(N(1 + \varphi_u)) = -c/\varepsilon \cdot \text{Kl}_{2^{2n}u}^{n,0}(\psi; k_E/k_F),$$

by Proposition 5.3, and this is not constant on  $u$  by Corollary A.5. This is a contradiction.

We next consider the case where  $\mathbf{x} = (\check{e}_1 + \cdots + \check{e}_n)/2$ . We take  $v \in k_E^\times$  such that  $\text{Nr}(v) = u$ , and set  $t = \text{diag}(v, \dots, v, c(v)^{-1}, \dots, c(v)^{-1}) \in H_{\mathbf{x}}$ . Then we have

$$tN(1 + \varphi_u)t^{-1} \equiv N(1 + \varphi_1) \pmod{H_{\mathbf{x}}^+},$$

hence

$$\Theta_{\pi_H}(N(1 + \varphi_u)) = \text{tr}(\dot{\rho}(N(1 + \varphi_1))).$$

Thus the character  $\Theta_{\pi_H}(N(1 + \varphi_u))$  is constant on  $u \in k_F^\times$ . This is a contradiction.  $\square$

## 5.5. Main theorem.

**Proposition 5.12.** *Let  $a \in k_F^\times$ ,  $\omega$  a conjugate self-dual character on  $k_E^\times$ , and  $\pi := \pi_{a, \zeta_\omega, \omega}$ . Then the representation  $\pi_H$  associated to  $\pi$  is given by  $\pi'_{a, \omega}$ . Here, we regard  $\omega$  also as a character on  $\text{U}(1)$  via the isomorphism  $z \mapsto z/c(z)$  from  $k_E^\times/k_F^\times$  to  $\text{U}(1)$ .*

*Proof.* By replacing the fixed uniformizer  $\varpi$ , we may assume that  $\pi = \pi_{1, \zeta_{\omega}, \omega}$ . By Proposition 5.11,  $\pi_H$  is simple supercuspidal. Let  $\pi_H$  be  $\pi'_{b, \omega'}$ , where  $b \in k_F^\times$  and  $\omega' \in \mathrm{U}(1)^*$ . Our task is to determine  $b$  and  $\omega'$ .

For  $u \in k_F^\times$ , we have

$$\Theta_{\pi_H}(N(1 + \varphi_u)) = c/\varepsilon \cdot \Theta_{\pi, \theta}(1 + \varphi_u) = -c/\varepsilon \cdot \mathrm{Kl}_{2^{2n}u}^{n,0}(\psi; k_E/k_F)$$

by Proposition 5.3. On the other hand, since  $N(1 + \varphi_u) \in I_H^+$  is an affine generic element with its affine simple components  $(2, \dots, 2, 2u)$ , we have

$$\Theta_{\pi_H}(N(1 + \varphi_u)) = -\mathrm{Kl}_{2^{2n}bu}^{n,0}(\psi; k_E/k_F)$$

by Proposition 3.18. Hence  $b = 1$  and  $c = \varepsilon$  by Proposition A.7.

For  $z \in Z(q)$ , we have

$$\Theta_{\pi_H}(z/c(z) \cdot N(1 + \varphi_u)) = \omega'(\overline{z/c(z)}) \cdot \Theta_{\pi_H}(N(1 + \varphi_u)).$$

From Proposition 5.3, we get  $\omega'(\overline{z/c(z)}) = \omega(\overline{z})$ .  $\square$

From this proposition and Corollary 5.5, for  $b \in k_F^\times$  and  $\omega' \in \mathrm{U}(1)^*$ , we know that the lifting of  $\pi'_{b, \omega'}$  to  $\mathrm{G}_{E/F}(2n)(F)$  via  $\xi_{+1}$  is given by  $\pi_{b, -\omega'(-1), \omega'}$ . Then, by Lemmas 4.14 and 4.15, we also know that the lifting of  $\pi'_{b, \omega'}$  to  $\mathrm{G}_{E/F}(2n)(F)$  via  $\xi_{-1}$  is given by  $\pi_{b, \omega'(-1), \omega'}$ .

In summary, we get the following result.

**Theorem 5.13** (Main theorem). *Let  $b \in k_F^\times$  and  $\omega' \in \mathrm{U}(1)^*$ . Under the parametrizations as in Sections 2.3 and 2.4, we have the following:*

- (1) *The  $L$ -packet containing the simple supercuspidal representation  $\pi'_{b, \omega'}$  of  $\mathrm{U}_{E/F}(2n)(F)$  is a singleton. In particular, the character of  $\pi'_{b, \omega'}$  is stable.*
- (2) *The lifting of the simple supercuspidal representation  $\pi'_{b, \omega'}$  of  $\mathrm{U}_{E/F}(2n)(F)$  to  $\mathrm{G}_{E/F}(2n)(F)$  via the  $L$ -embedding  $\xi_\kappa$  is again simple supercuspidal, and given by  $\pi_{b, -\kappa\omega'(-1), \omega'}$ .*

*Remark 5.14.* From this result, we know that the  $L$ -parameter of  $\pi'_{b, \omega'}$  is equal to that of  $\pi_{b, -\omega'(-1), \omega'}$ . On the other hand,  $L$ -parameters of simple supercuspidal representations of general linear groups have been determined explicitly by the works [BH05] and [IT15]. Therefore we can get an explicit description of  $L$ -parameters of simple supercuspidal representations of  $\mathrm{U}_{E/F}(2n)(F)$ .

## 6. MAIN THEOREM: THE ODD CASE

In this section we treat the odd case. Let  $G := \mathrm{G}_{E/F}(2n+1)$ , and  $H := \mathrm{U}_{E/F}(2n+1)$ .

**6.1. Norms of  $1 + \varphi_{\epsilon^{-1}u}$  and  $\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})$ .** For  $u \in k_F^\times$ , we consider

$$1 + \varphi_{\epsilon^{-1}u} \in I^+ \subset G(F), \text{ and}$$

$$N(1 + \varphi_{\epsilon^{-1}u}) = (1 + \varphi_{\epsilon^{-1}u})\theta(1 + \varphi_{\epsilon^{-1}u}) \in I_H^+ \subset H(F).$$

These are affine generic elements.

**Proposition 6.1.** *The element  $1 + \varphi_{\epsilon^{-1}u} \in G(F)$  is strongly  $\theta$ -regular  $\theta$ -semisimple and  $N(1 + \varphi_{\epsilon^{-1}u}) \in H(F)$  is strongly regular semisimple. Moreover,  $N(1 + \varphi_{\epsilon^{-1}u})$  is a norm of  $1 + \varphi_{\epsilon^{-1}u}$ .*

*Proof.* We first show that  $1 + \varphi_{\epsilon^{-1}u}$  is  $\theta$ -semisimple. By Lemma 4.1,  $1 + \varphi_{\epsilon^{-1}u}$  is semisimple. Since

$$\begin{aligned}\theta(1 + \varphi_{\epsilon^{-1}u}) &= J(1 + {}^t\varphi_{-\epsilon^{-1}u})^{-1}J^{-1} \\ &= J(1 - {}^t\varphi_{-\epsilon^{-1}u} + {}^t\varphi_{-\epsilon^{-1}u}^2 - \cdots)J^{-1} \\ &= 1 + \varphi_{\epsilon^{-1}u} + \varphi_{\epsilon^{-1}u}^2 + \cdots,\end{aligned}$$

$1 + \varphi_{\epsilon^{-1}u}$  commutes with  $\theta(1 + \varphi_{\epsilon^{-1}u})$ . Moreover, since the residual characteristic is not equal to 2,  $N(1 + \varphi_{\epsilon^{-1}u})$  is an affine generic element of  $G(F)$ , hence regular semisimple by Lemma 4.1. Therefore  $1 + \varphi_{\epsilon^{-1}u}$  is  $\theta$ -semisimple by Lemma 4.3.

By the definition of  $\mathcal{A}_{H/G}$ ,  $N(1 + \varphi_{\epsilon^{-1}u})$  corresponds to  $1 + \varphi_{\epsilon^{-1}u}$  via  $\mathcal{A}_{H/G}$ . Then  $1 + \varphi_{\epsilon^{-1}u}$  is strongly  $\theta$ -regular and  $N(1 + \varphi_{\epsilon^{-1}u})$  is strongly regular by Lemma 4.4.  $\square$

Next, for  $u \in k_F^\times$  we consider the elements

$$\begin{aligned}\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u}) &\in G(F), \text{ and} \\ N(\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})) &= (\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})) \cdot \theta(\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})) \\ &= -N(1 + \varphi_{\epsilon^{-1}u}) \in H(F).\end{aligned}$$

**Proposition 6.2.** *The element  $\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u}) \in G(F)$  is strongly  $\theta$ -regular  $\theta$ -semisimple and  $-N(1 + \varphi_{\epsilon^{-1}u}) \in H(F)$  is strongly regular semisimple. Moreover,  $-N(1 + \varphi_{\epsilon^{-1}u})$  is a norm of  $\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})$ .*

*Proof.* By the same argument in the proof of Proposition 6.1, we only have to show that  $\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})$  is  $\theta$ -semisimple. Since the characteristic polynomial of  $\varphi_{\epsilon^{-1}u}$  is  $t^{2n} - \varpi\epsilon^{-1}u$  and irreducible over  $E$ ,  $\varphi_{\epsilon^{-1}u}$  is semisimple. Hence so is  $\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})$ . Since

$$\begin{aligned}\theta(\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})) &= -\varphi_{\epsilon^{-1}u}^{-1}J(1 - {}^t\varphi_{-\epsilon^{-1}u} + {}^t\varphi_{-\epsilon^{-1}u}^2 - \cdots)J^{-1} \\ &= -\varphi_{\epsilon^{-1}u}^{-1}(1 + \varphi_{\epsilon^{-1}u} + \varphi_{\epsilon^{-1}u}^2 + \cdots),\end{aligned}$$

$\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})$  commutes with  $\theta(\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u}))$ . Finally, as

$$N(\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})) = -N(1 + \varphi_{\epsilon^{-1}u})$$

is regular semisimple by the proof of Proposition 6.1,  $\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})$  is  $\theta$ -semisimple by Lemma 4.3.  $\square$

**6.2. Parity of simple supercuspidal representations.** Let  $a \in k_F^\times$ . From Lemmas 4.14 and 4.15, we know that the parity of the  $L$ -parameter of  $\pi_{a\epsilon, 1, \omega}$  is different from that of  $\pi_{a\epsilon, -1, \omega}$ . In this subsection, we determine the parity of these parameters. We first consider the case where  $a = 1$ .

Let  $\zeta_\omega \in \{\pm 1\}$  such that the parity of the  $L$ -parameter of  $\pi_{\epsilon, \zeta_\omega, \omega}$  is 1. We put  $\pi := \pi_{\epsilon, \zeta_\omega, \omega}$ . Then the  $L$ -parameter of  $\pi$  factors through the standard base change embedding  $\xi_{+1}$  by Lemma 4.14. Let  $\pi_H$  be the supercuspidal representation of  $H(F)$  which is associated to  $\pi$ .

**Proposition 6.3.** *For  $u \in k_F^\times$  and  $z \in \mathcal{O}_E^\times$ , we have*

$$\Theta_{\pi_H}(z/c(z) \cdot N(1 + \varphi_{\epsilon^{-1}u})) = c/\varepsilon \cdot \Theta_{\pi, \theta}(z(1 + \varphi_{\epsilon^{-1}u})) = c/\varepsilon \cdot \omega(\overline{z}) \cdot \text{Kl}_{2^{2n+1}u}^{n,1}(\psi; k_E/k_F).$$



*Proof.* We write  $g$  and  $h$  for  $1 + \varphi_{\epsilon^{-1}u}$  and  $N(1 + \varphi_{\epsilon^{-1}u})$ , respectively. Then, by Proposition 6.1 and Lemma 4.2,  $z/c(z) \cdot h \in H^{\text{srs}}(F)$  is a norm of  $zg \in G^{\theta\text{-srs}}(F)$ . Therefore we have  $c/\varepsilon \cdot \Theta_{\pi,\theta}(zg) = \Theta_{\pi_H}(z/c(z) \cdot h)$  by Corollary 4.19.

On the other hand, since we have  $\omega_{\pi}(z) = \omega(\bar{z})$ , where  $\omega_{\pi}$  is the central character of  $\pi$ ,

$$\Theta_{\pi,\theta}(zg) = \omega(\bar{z}) \cdot \Theta_{\pi,\theta}(g).$$

Since  $N(g)$  is affine generic,  $g$  satisfies the assumption of Proposition 3.13. As the affine simple components of  $g = 1 + \varphi_{\epsilon^{-1}u}$  is  $(1, \dots, 1, \epsilon^{-1}u)$ , we have

$$\Theta_{\pi,\theta}(g) = \text{Kl}_{2^{2n+1}u}^{n,1}(\psi; k_E/k_F).$$

□

**Proposition 6.4.** *For  $u \in k_F^{\times}$ , we have*

$$\Theta_{\pi_H}(-N(1 + \varphi_{\epsilon^{-1}u})) = c/\varepsilon \cdot \Theta_{\pi,\theta}(\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})) = c/\varepsilon \cdot \zeta_{\omega} \cdot \text{Kl}_{2^{2n+1}u}^{n,1}(\psi; k_E/k_F).$$

*Proof.* We write  $g'$  and  $h'$  for  $\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})$  and  $-N(1 + \varphi_{\epsilon^{-1}u})$ , respectively. Then, by Proposition 6.2,  $h' \in H^{\text{srs}}(F)$  is a norm of  $g' \in G^{\theta\text{-srs}}(F)$ . Therefore we have  $c/\varepsilon \cdot \Theta_{\pi,\theta}(g') = \Theta_{\pi_H}(h')$  by Corollary 4.19.

Since  $-N(g') = N(1 + \varphi_u)$  is affine generic,  $1 + \varphi_{\epsilon^{-1}u}$  satisfies the assumption of Proposition 3.16. As the affine simple components of  $1 + \varphi_{\epsilon^{-1}u}$  is  $(1, \dots, 1, \epsilon^{-1}u)$ , we have

$$\Theta_{\pi,\theta}(\varphi_{\epsilon^{-1}u}(1 + \varphi_{\epsilon^{-1}u})) = \zeta_{\omega} \cdot \text{Kl}_{2^{2n+1}u}^{n,1}(\psi; k_E/k_F).$$

□

**Corollary 6.5.** *We have  $\zeta_{\omega} = \omega(\epsilon)$ .*

*Proof.* We regard  $\epsilon$  as an element of  $\mathcal{O}_E^{\times}$  by the Teichmüller lift. By Propositions 6.3 and 6.4, we have

$$\omega(\epsilon) \cdot \text{Kl}_{2^{2n+1}u}^{n,1}(\psi; k_E/k_F) = \zeta_{\omega} \cdot \text{Kl}_{2^{2n+1}u}^{n,1}(\psi; k_E/k_F).$$

Since there exists  $u \in k_F^{\times}$  such that  $\text{Kl}_{2^{2n+1}u}^{n,1}(\psi; k_E/k_F) \neq 0$  by Corollary A.5, we get  $\zeta_{\omega} = \omega(\epsilon)$ . □

Note that  $\zeta_{\omega}$  depends only on the character  $\omega$ . Thus the parity of the  $L$ -parameter of  $\pi_{a\epsilon, \zeta_{\omega}, \omega}$  is 1 for any  $a \in k_F^{\times}$ .

**6.3. Existence of  $I_H^{++}$ -fixed vector.** Let  $\pi := \pi_{\epsilon, \zeta_{\omega}, \omega}$  and  $\pi_H$  the supercuspidal representation of  $H(F)$  associated to  $\pi$ .

**Lemma 6.6.** *Let  $h \in H(F)$  be an affine generic element with its affine simple components  $(h_1, \dots, h_n, h_{2n+1})$ . Then  $\Theta_{\pi_H}(h)$  is equal to either 0 or*

$$c/\varepsilon \cdot \text{Kl}_{\text{Nr}(h_1) \cdots \text{Nr}(h_n) h_{2n+1}\epsilon}^{n,1}(\psi; k_E/k_F).$$

*Proof.* For such  $h$ , we take  $g \in G(F)$  satisfying the conditions in Proposition 4.6. Since  $h \in H^{\text{srs}}(F)$ ,  $g \in G^{\theta\text{-srs}}(F)$ , and  $h$  is a norm of  $g$ , we have

$$c/\varepsilon \cdot \Theta_{\pi,\theta}(g) = \Theta_{\pi_H}(h)$$

by Corollary 4.19. We compute the left-hand side of this equality.

If there is not  $x \in G(F)$  such that  $xg\theta(x)^{-1} \in ZI^+\langle\varphi_{\epsilon^{-1}}\rangle$ , then  $\Theta_{\pi,\theta}(g) = 0$  by the twisted character formula (Theorem 3.4).

Let us consider the case where there exists  $x \in G(F)$  such that  $xg\theta(x)^{-1} \in ZI^+\langle\varphi_{\epsilon^{-1}}\rangle$ . By Lemma 3.11, we may assume  $x \in Z_S(q)$ . Since  $\theta(\varphi_{\epsilon^{-1}}) = -\varphi_{\epsilon^{-1}}^{-1}$  and  $\varphi_{\epsilon^{-1}}$  normalizes  $I^+$ , we have

$$\varphi_{\epsilon^{-1}}^k xg\theta(\varphi_{\epsilon^{-1}}^k x)^{-1} \in ZI^+ \amalg ZI^+ \varphi_{\epsilon^{-1}}$$

for some  $k \in \mathbb{Z}$ .

We first consider the case where  $\varphi_{\epsilon^{-1}}^k xg\theta(\varphi_{\epsilon^{-1}}^k x)^{-1} \in ZI^+$ . We take  $z \in Z$  and  $g_0 \in I^+$  such that  $\varphi_{\epsilon^{-1}}^k xg\theta(\varphi_{\epsilon^{-1}}^k x)^{-1} = zg_0$ . Then we have  $z/c(z) \in 1 + \mathfrak{p}_E$  and  $\omega_\pi(z) = 1$  by the same argument in the proof of Lemma 5.6 (note that  $\omega_\pi(\varpi) = 1$  since  $\varpi\epsilon^{-1}I_{2n+1} = \varphi_{\epsilon^{-1}}^{2n+1}$  and  $\chi_{\epsilon, \zeta_\omega, \omega}(\varphi_{\epsilon^{-1}}) = \zeta_\omega = \omega(\epsilon)$ ).

Let  $x = \text{diag}(t_1, \dots, t_{2n+1})$ . Let  $(g_1, \dots, g_{2n+1})$  be the affine simple components of  $g_0$ . Then the affine simple components of  $N(zg_0) = z/c(z) \cdot N(g_0)$  are

$$(g_1 + c(g_{2n}), \dots, g_{2n} + c(g_1), g_{2n+1} - c(g_{2n+1})).$$

On the other hand, since the affine simple components of  $N(g) = h$  are

$$(h_1, \dots, h_n, c(h_n), \dots, c(h_1), h_{2n+1}),$$

those of  $N(zg_0) = \varphi_{\epsilon^{-1}}^k xN(g)(\varphi_{\epsilon^{-1}}^k x)^{-1}$  are products of  $(1, \dots, 1, \epsilon^{-1})$  and some cyclic permutation of

$$\left( \frac{t_1}{t_2} h_1, \dots, \frac{t_{2n}}{t_{2n+1}} c(h_1), \epsilon \frac{t_{2n+1}}{t_1} h_{2n+1} \right).$$

Therefore  $g_0$  satisfies the assumption of Proposition 3.13, and we have

$$\begin{aligned} \Theta_{\pi, \theta}(g) &= \Theta_{\pi, \theta}(\varphi_{\epsilon^{-1}}^k xg\theta(\varphi_{\epsilon^{-1}}^k x)^{-1}) \\ &= \Theta_{\pi, \theta}(zg_0) \\ &= \omega_\pi(z) \Theta_{\pi, \theta}(g_0) \\ &= \text{KL}_{\text{Nr}(g_1 + c(g_{2n})) \cdots \text{Nr}(g_n + c(g_{n+1})) \text{Tr}(\epsilon g_{2n+1})}^{n, 1}(\psi; k_E/k_F) \\ &= \text{KL}_{\text{Nr}(h_1) \cdots \text{Nr}(h_n) h_{2n+1} \epsilon}^{n, 1}(\psi; k_E/k_F). \end{aligned}$$

We next consider the case where  $\varphi_{\epsilon^{-1}}^k xg\theta(\varphi_{\epsilon^{-1}}^k x)^{-1} \in ZI^+ \varphi_{\epsilon^{-1}}$ . We take  $z \in Z$  and  $g_0 \in I^+$  such that  $\varphi_{\epsilon^{-1}}^k xg\theta(\varphi_{\epsilon^{-1}}^k x)^{-1} = z\varphi_{\epsilon^{-1}} g_0$ . Then we have  $z/c(z) \in -(1 + \mathfrak{p}_E)$  and  $\omega_\pi(z) = \omega(\epsilon)$  by the same argument in the proof of Lemma 5.6.

Let  $x = \text{diag}(t_1, \dots, t_{2n+1})$ . Let  $(g_1, \dots, g_{2n+1})$  be the affine simple components of  $g_0$ . Then the affine simple components of  $N(z\varphi_{\epsilon^{-1}} g_0) = z/c(z) \cdot N(\varphi_{\epsilon^{-1}} g_0)$  are

$$(g_2 + c(g_{2n}), \dots, g_{2n} + c(g_2), \epsilon g_{2n+1} + c(g_1), \epsilon^{-1} g_1 - c(g_{2n+1})).$$

On the other hand, since the affine simple components of  $N(g) = h$  are

$$(h_1, \dots, h_n, c(h_n), \dots, c(h_1), h_{2n+1}),$$

those of  $N(z\varphi_{\epsilon^{-1}} g_0) = \varphi_{\epsilon^{-1}}^k xN(g)(\varphi_{\epsilon^{-1}}^k x)^{-1}$  are products of  $(1, \dots, 1, \epsilon^{-1})$  and some cyclic permutation of

$$\left( \frac{t_1}{t_2} h_1, \dots, \frac{t_{2n}}{t_{2n+1}} c(h_1), \epsilon \frac{t_{2n+1}}{t_1} h_{2n+1} \right).$$

Therefore  $\varphi_{\epsilon^{-1}}g_0$  satisfies the assumption of Proposition 3.16, and we have

$$\begin{aligned}
\Theta_{\pi,\theta}(g) &= \Theta_{\pi,\theta}(z\varphi_{\epsilon^{-1}}g_0) \\
&= \omega_{\pi}(z)\Theta_{\pi,\theta}(\varphi_{\epsilon^{-1}}g_0) \\
&= \omega_{\pi}(\epsilon) \cdot \zeta_{\omega} \cdot \text{Kl}_{\text{Nr}(g_1 - \epsilon c(g_{2n+1})) \text{Nr}(g_2 + c(g_{2n})) \cdots \text{Nr}(g_n + c(g_{n+2})) \text{Tr}(g_{n+1})}^{n,1}(\psi; k_E/k_F) \\
&= \text{Kl}_{\text{Nr}(h_1) \cdots \text{Nr}(h_n) h_{2n+1}\epsilon}^{n,1}(\psi; k_E/k_F).
\end{aligned}$$

Here, we used Corollary 6.5 in the last equality.  $\square$

As in Corollary 5.7, we get the following result by Lemma 6.6

**Corollary 6.7.** *The representation  $(\pi_H, V_H)$  has a nonzero  $I_H^{++}$ -fixed vector.*

**6.4. Simple supercuspidality of  $\pi_H$ .** Let  $\pi$  and  $\pi_H$  be as in the previous subsection. In this subsection, we prove that  $\pi_H$  is simple supercuspidal.

By Corollary 6.7, the finite abelian group  $I_H^+/I_H^{++}$  acts on the nonzero subspace  $\pi_H^{I_H^{++}}$  of  $\pi_H$ , and it decomposes into a direct sum of characters of  $I_H^+/I_H^{++}$ . We take a character  $\eta$  of  $I_H^+$  which is contained in it.

By the same argument in the proof of Lemma 5.8, we get the following lemma.

**Lemma 6.8.** *If  $\eta$  is an affine generic character of  $I_H^+$ , then  $\pi_H$  is simple supercuspidal.*

From this lemma, we are reduced to prove the affine genericity of  $\eta$ . We first prove two technical lemmas which will be needed in the proof of the affine genericity of  $\eta$ .

**Lemma 6.9.** *Let  $\beta \in \Pi_H$  be a simple affine root of  $S_H$  in  $H$ . Then the subgroup  $\langle I_H^{++}, U_{\beta} \rangle$  of  $H(F)$  contains  $H_{\mathbf{x}}^+$  for some point  $\mathbf{x}$  of the apartment  $\mathcal{A}(H, S_H)$ , where  $H_{\mathbf{x}}^+$  is the pro-unipotent radical of the parahoric subgroup of  $H(F)$  associated to  $\mathbf{x}$ .*

*Proof.* As in the even case, we are reduced to find a point  $\mathbf{x}$  satisfying the following condition:

(\*) If an affine root  $\alpha$  satisfies  $\alpha(\mathbf{x}) > 0$ , then we have  $\alpha \in (\Psi_H^+ \setminus \Pi_H) \cup \{\beta\}$ .

We first consider the case where  $\beta = -2e_1 + 1$ . In this case, we take  $\mathbf{x} = 0$ . If  $\alpha = a + r \in \Psi_H$  satisfies  $\alpha(\mathbf{x}) > 0$ , then we have  $r > 0$ . Hence  $\alpha \in \Psi_H^+ \setminus \Pi_H$  or  $\alpha = \beta$ .

We next consider the other cases. We define a point  $\mathbf{x} \in \mathcal{A}(H, S_H)$  as follows:

$$\mathbf{x} := \begin{cases} (\check{e}_1 + \cdots + \check{e}_i)/2 & (\beta = e_i - e_{i+1} \text{ for } 1 \leq i < n), \\ (\check{e}_1 + \cdots + \check{e}_n)/2 & (\beta = e_n). \end{cases}$$

It is enough to prove (\*) for  $\alpha = a + r_a \in \Psi_H$ , where  $r_a \in \mathbb{Z}$  is the smallest integer satisfying  $a(\mathbf{x}) + r_a > 0$ .

- If  $a \in \Phi_H^+ \setminus \Delta_H$ , then we have  $0 \leq a(\mathbf{x}) \leq 1$ . Hence  $\alpha(\mathbf{x}) > 0$  implies that  $r_a \geq 0$ . Therefore  $\alpha = a + r_a$  belongs to  $\Psi_H^+ \setminus \Pi_H$ .
- If  $a \in \Delta_H$ , then we have

$$a(\mathbf{x}) = \begin{cases} 0 & (a \neq \beta), \\ 1/2 & (a = \beta). \end{cases}$$

Hence we have

$$\alpha(\mathbf{x}) > 0 \iff \begin{cases} r_a > 0 & (a \neq \beta), \\ r_a \geq 0 & (a = \beta). \end{cases}$$

Therefore we have either  $\alpha \in \Psi_H^+ \setminus \Pi_H$  or  $\alpha = \beta$ .

- If  $a \in \Phi_H^- \setminus \{-2e_1\}$ , then we have  $-1 \leq a(\mathbf{x}) \leq 0$ . Hence  $\alpha(\mathbf{x}) > 0$  implies that  $r_a \geq 1$ . Therefore  $\alpha = a + r_a$  belongs to  $\Psi_H^+ \setminus \Pi_H$ .
- If  $a = -2e_1$ , then we have  $\alpha(\mathbf{x}) = r_a - 1$ . Hence  $\alpha(\mathbf{x}) > 0$  implies that  $r_a \geq 2$ , and we have  $\alpha \in \Psi_H^+ \setminus \Pi_H$ .

□

**Lemma 6.10.** *Let  $\mathbf{x} = 0$  be a hyperspecial point in the apartment  $\mathcal{A}(H, S_H)$ , and  $H_0$  the hyperspecial subgroup of  $H(F)$  associated to  $\mathbf{x} = 0$ . Let  $y \in H(F)$ . If  $y$  satisfies  $yhy^{-1} \in H_0$  for an affine generic element  $h \in H(F)$ , then  $y \in H_0$ .*

*Proof.* Let  $y \in H(F)$  satisfying  $yhy^{-1} \in H_0$  for an affine generic element  $h \in I_H^+$ . As affine genericity is preserved by  $I_H^+$ -conjugation, any element of  $H_0 y I_H^+$  satisfies the same condition as  $y$ . It suffices to show  $H_0 y I_H^+ \subset H_0$ .

The isomorphism in Proposition 2.3 induces an isomorphism

$$H_0 \backslash H(F) / I_H^+ \cong ((N_{S_H}(F) \cap H_0) / (Z_{S_H})_1) \backslash (N_{S_H}(F) / (Z_{S_H})_1)$$

(see [Ric13, Theorem 1.4]) and the right-hand side is represented by

$$Z_{S_H}(\varpi^{\mathbb{Z}}) := \{ \text{diag}(\varpi^{r_1}, \dots, \varpi^{r_n}, 1, \varpi^{-r_n}, \dots, \varpi^{-r_1}) \mid r_1, \dots, r_n \in \mathbb{Z} \}.$$

Hence we may assume

$$\begin{aligned} y &= \text{diag}(t_1, \dots, t_{2n+1}) \\ &= \text{diag}(\varpi^{r_1}, \dots, \varpi^{r_n}, 1, \varpi^{-r_n}, \dots, \varpi^{-r_1}) \in Z_{S_H}(\varpi^{\mathbb{Z}}). \end{aligned}$$

Since  $(yhy^{-1})_{ij} = t_i/t_j \cdot h_{ij}$ , we have the following inequalities:

$$\begin{aligned} r_1 - r_2 &\geq 0, \\ &\vdots \\ r_{n-1} - r_n &\geq 0, \\ r_n &\geq 0, \text{ and} \\ -2r_1 &\geq -1. \end{aligned}$$

Therefore  $(r_1, \dots, r_n)$  is  $(0, \dots, 0)$ .

Hence we have  $H_0 y I_H^+ = H_0 I_H^+ = H_0$ , and this completes the proof. □

**Proposition 6.11.** *Every character  $\eta$  of  $I_H^+$  which is contained in  $\pi_H^{I_H^{++}}$  is affine generic. In particular, the representation  $\pi_H$  is simple supercuspidal.*

*Proof.* We suppose that  $\eta$  is not affine generic.

By Lemma 6.9 and the same argument in the proof of Proposition 5.11, we know that  $\pi_H$  can be obtained by the compact induction of a representation of the reduction of a hyperspecial subgroup  $H_{\mathbf{x}}$  of  $H(F)$ . Since every hyperspecial vertex of the building  $\mathcal{B}(H)$  of  $H$  is conjugate to 0, we may assume  $\mathbf{x} = 0$  (see [Tit79, Section 4]).

Let  $H_0^+$  be the pro-unipotent radical of  $H_0$ . Let  $\pi_H \cong \text{c-Ind}_{H_0}^{H(F)} \dot{\rho}$ , where  $\dot{\rho}$  is the inflation of a representation  $\rho$  of  $H_0/H_0^+ \cong \text{U}_{k_E/k_F}(2n+1)(k_F)$  to  $H_0$ . Then, by the character formula (Theorem 3.2), we have

$$\Theta_{\pi_H}(N(1 + \varphi_{\epsilon^{-1}u})) = \sum_{\substack{y \in H_0 \setminus H(F) \\ yN(1 + \varphi_{\epsilon^{-1}u})y^{-1} \in H_0}} \text{tr}(\dot{\rho}(yN(1 + \varphi_{\epsilon^{-1}u})y^{-1}))$$

for any  $u \in k_F^\times$ . By Lemma 6.10, we have

$$\Theta_{\pi_H}(N(1 + \varphi_{\epsilon^{-1}u})) = \text{tr}(\dot{\rho}(N(1 + \varphi_{\epsilon^{-1}u}))).$$

Then the image of  $N(1 + \varphi_{\epsilon^{-1}u})$  in  $H_0/H_0^+$  is independent of  $u$ . Hence the character  $\Theta_{\pi_H}(N(1 + \varphi_{\epsilon^{-1}u}))$  is constant on  $u \in k_F^\times$ .

However we have

$$\Theta_{\pi_H}(N(1 + \varphi_{\epsilon^{-1}u})) = c/\varepsilon \cdot \text{Kl}_{2^{2n+1}u}^{n,1}(\psi; k_E/k_F),$$

by Proposition 6.3, and this is not constant on  $u$  by Corollary A.5. This is a contradiction.  $\square$

### 6.5. Main theorem.

**Proposition 6.12.** *Let  $a \in k_F^\times$ ,  $\omega$  a conjugate self-dual character on  $k_E^\times$ , and  $\pi := \pi_{a\varepsilon, \zeta_\omega, \omega}$ . Then the representation  $\pi_H$  associated to  $\pi$  is given by  $\pi'_{a, \omega}$ . Here, we regard  $\omega$  also as a character on  $\text{U}(1)$  via the isomorphism  $z \mapsto z/c(z)$  from  $k_E^\times/k_F^\times$  to  $\text{U}(1)$ .*

*Proof.* By replacing the fixed uniformizer  $\varpi$ , we may assume that  $\pi = \pi_{\varepsilon, \zeta_\omega, \omega}$ . By Proposition 6.11,  $\pi_H$  is simple supercuspidal. Let  $\pi_H$  be  $\pi'_{b, \omega'}$ , where  $b \in k_F^\times$  and  $\omega' \in \text{U}(1)^*$ . Our task is to determine  $b$  and  $\omega'$ .

For  $u \in k_F^\times$ , we have

$$\Theta_{\pi_H}(N(1 + \varphi_{\epsilon^{-1}u})) = c/\varepsilon \cdot \Theta_{\pi, \theta}(1 + \varphi_{\epsilon^{-1}u}) = c/\varepsilon \cdot \text{Kl}_{2^{2n+1}u}^{n,1}(\psi; k_E/k_F)$$

by Proposition 6.3. On the other hand, since  $N(1 + \varphi_{\epsilon^{-1}u}) \in I_H^+$  is an affine generic element with its affine simple components  $(2, \dots, 2, 2\epsilon^{-1}u)$ , we have

$$\Theta_{\pi_H}(N(1 + \varphi_{\epsilon^{-1}u})) = \text{Kl}_{2^{2n+1}bu}^{n,1}(\psi; k_E/k_F)$$

by Proposition 3.20. Hence  $b = 1$  and  $c = \varepsilon$  by Proposition A.7.

For  $z \in Z(q)$ , we have

$$\Theta_{\pi_H}(z/c(z) \cdot N(1 + \varphi_{\epsilon^{-1}u})) = \omega'(\overline{z/c(z)}) \cdot \Theta_{\pi_H}(N(1 + \varphi_{\epsilon^{-1}u})).$$

From Proposition 6.3, we get  $\omega'(\overline{z/c(z)}) = \omega(\overline{z})$ .  $\square$

From this proposition and Corollary 6.5, for  $b \in k_F^\times$  and  $\omega' \in \text{U}(1)^*$ , we know that the lifting of  $\pi'_{b, \omega'}$  to  $\text{G}_{E/F}(2n+1)(F)$  via  $\xi_{+1}$  is given by  $\pi_{b\varepsilon, \omega'(-1), \omega'}$ . Then, by Lemmas 4.14 and 4.15, we also know that the lifting of  $\pi'_{b, \omega'}$  to  $\text{G}_{E/F}(2n+1)(F)$  via  $\xi_{-1}$  is given by  $\pi_{b\varepsilon, -\omega'(-1), \omega'}$ .

In summary, we get the following result.

**Theorem 6.13** (Main theorem). *Let  $b \in k_F^\times$  and  $\omega' \in \text{U}(1)^*$ . Under the parametrizations as in Sections 2.3 and 2.5, we have the following:*

- (1) The  $L$ -packet containing the simple supercuspidal representation  $\pi'_{b,\omega'}$  of  $U_{E/F}(2n+1)(F)$  is a singleton. In particular, the character of  $\pi'_{b,\omega'}$  is stable.
- (2) The lifting of the simple supercuspidal representation  $\pi'_{b,\omega'}$  of  $U_{E/F}(2n+1)(F)$  to  $G_{E/F}(2n+1)(F)$  via the  $L$ -embedding  $\xi_\kappa$  is again simple supercuspidal, and given by  $\pi_{b\epsilon,\kappa\omega'(-1),\omega'}$ .

*Remark 6.14.* As in Remark 5.14, we can determine the  $L$ -parameters of simple supercuspidal representations of  $U_{E/F}(2n+1)(F)$  from this result.

#### APPENDIX A. FOURIER TRANSFORM OF THE KLOOSTERMAN SUM

**Definition A.1.** Let  $k$  be a finite field. Let  $\chi$  be a multiplicative character of  $k^\times$  and  $\psi$  a nontrivial additive character of  $k$ . Then we define the Gauss sum with respect to  $(k, \chi, \psi)$  by

$$G(k; \chi, \psi) := \sum_{t \in k^\times} \chi(t) \psi(t).$$

Let  $k_E/k_F$  be a finite extension of finite fields of degree  $r$ . Then we have the following classical result.

**Fact A.2** (Hasse-Davenport relation, [Del77, Sommes trig. Théorème 1.15]). *Let  $\chi$  be a multiplicative character of  $k_F^\times$  and  $\psi$  a nontrivial additive character of  $k_F$ . Then we have*

$$G(k_E; \chi \circ \text{Nr}, \psi \circ \text{Tr}) = (-1)^{r-1} G(k_F; \chi, \psi)^r,$$

where  $\text{Nr} := \text{Nr}_{k_E/k_F}$  and  $\text{Tr} := \text{Tr}_{k_E/k_F}$ .

**Definition A.3** (Kloosterman sum). Let  $\psi: k_F \rightarrow \mathbb{C}^\times$  be a nontrivial additive character,  $a$  an element of  $k_F^\times$ , and  $n, m$  non-negative integers. We define the Kloosterman sum with respect to  $(\psi, k_E/k_F, a, n, m)$  by

$$\text{Kl}_a^{n,m}(\psi; k_E/k_F) := \sum_{\substack{t_1, \dots, t_n \in k_E \\ s_1, \dots, s_m \in k_F \\ \text{Nr}(t_1) \cdots \text{Nr}(t_n) s_1 \cdots s_m = a}} \psi \circ \text{Tr}(t_1 + \cdots + t_n) \cdot \psi(s_1 + \cdots + s_m).$$

**Proposition A.4.** *Let  $\chi$  be a multiplicative character of  $k_F^\times$ . Then we have*

$$\sum_{a \in k_F^\times} \chi(a) \text{Kl}_a^{n,m}(\psi; k_E/k_F) = G(k_E; \chi \circ \text{Nr}, \psi \circ \text{Tr})^n \cdot G(k_F; \chi, \psi)^m.$$

*Proof.* We can compute as follows:

$$\begin{aligned} \text{LHS} &= \sum_{\substack{t_1, \dots, t_n \in k_E^\times \\ s_1, \dots, s_m \in k_F^\times}} \chi(\text{Nr}(t_1) \cdots \text{Nr}(t_n) s_1 \cdots s_m) \cdot \psi \circ \text{Tr}(t_1 + \cdots + t_n) \cdot \psi(s_1 + \cdots + s_m) \\ &= \left( \sum_{t \in k_E^\times} \chi \circ \text{Nr}(t) \cdot \psi \circ \text{Tr}(t) \right)^n \cdot \left( \sum_{s \in k_F^\times} \chi(s) \psi(s) \right)^m = \text{RHS}. \end{aligned}$$

□

**Corollary A.5.** (1) *The sum  $\text{Kl}_a^{n,m}(\psi; k_E/k_F)$  is not constant on  $a \in k_F^\times$ .*  
(2) *For some  $a \in k_F^\times$ ,  $\text{Kl}_a^{n,m}(\psi; k_E/k_F) \neq 0$*

*Proof.* Assume that the sum  $\text{Kl}_a^{n,m}(\psi; k_E/k_F)$  is constant on  $a \in k_F^\times$ . If we take a nontrivial multiplicative character  $\chi$  of  $k_F^\times$ , then by Proposition A.4 we have

$$G(k_E; \chi \circ \text{Nr}, \psi \circ \text{Tr})^n \cdot G(k_F; \chi, \psi)^m = 0.$$

However this contradicts the non-zerosness of the Gauss sum.  $\square$

**Corollary A.6.** *For  $n \in \mathbb{Z}_{\geq 0}$  and  $a \in k_F^\times$ , we have*

$$\text{Kl}_a^{n,r}(\psi; k_E/k_F) = (-1)^{r-1} \text{Kl}_a^{n+1,0}(\psi; k_E/k_F).$$

*Proof.* It suffices to show the equality for their Fourier transforms. Let  $\chi$  be a multiplicative character of  $k_F^\times$ . By Proposition A.4, we have

$$\sum_{a \in k_F^\times} \chi(a) \text{Kl}_a^{n,r}(\psi; k_E/k_F) = G(k_E; \chi \circ \text{Nr}, \psi \circ \text{Tr})^n \cdot G(k_F; \chi, \psi)^r.$$

On the other hand, we have

$$(-1)^{r-1} \sum_{a \in k_F^\times} \chi(a) \text{Kl}_a^{n+1,0}(\psi; k_E/k_F) = (-1)^{r-1} G(k_E; \chi \circ \text{Nr}, \psi \circ \text{Tr})^{n+1}.$$

Therefore we get the conclusion by the Hasse-Davenport relation (Fact A.2).  $\square$

**Proposition A.7.** *Let  $a, b \in k_F^\times$  and  $c \in \mathbb{C}^\times$ .*

*If*

$$\text{Kl}_{ta}^{n,m}(\psi; k_E/k_F) = c \cdot \text{Kl}_{tb}^{n,m}(\psi; k_E/k_F)$$

*for every  $t \in k_F^\times$ , then  $c = 1$  and  $a = b$ .*

*Proof.* By summing up over  $t \in k_F^\times$  and using Proposition A.4, we get  $(-1)^{n+m} = c \cdot (-1)^{n+m}$ , therefore  $c = 1$ .

We next show  $a = b$ . We may assume  $b = 1$ . It suffices to show that if  $a \neq 1$ , then there exists  $t \in k_F^\times$  such that  $\text{Kl}_{ta}^{n,m}(\psi; k_E/k_F) \neq \text{Kl}_t^{n,m}(\psi; k_E/k_F)$ . Let us assume  $a \neq 1$ , then we can take a multiplicative character  $\chi$  of  $k_F^\times$  satisfying  $\chi(a) \neq 1$ . Then we have

$$\begin{aligned} & \sum_{t \in k_F^\times} \chi(t) (\text{Kl}_{ta}^{n,m}(\psi; k_E/k_F) - \text{Kl}_t^{n,m}(\psi; k_E/k_F)) \\ &= (\chi(a)^{-1} - 1) \sum_{t \in k_F^\times} \chi(t) \text{Kl}_t^{n,m}(\psi; k_E/k_F) \\ &= (\chi(a)^{-1} - 1) G(k_E; \chi \circ \text{Nr}, \psi \circ \text{Tr})^n \cdot G(k_F; \chi, \psi)^m \\ &\neq 0. \end{aligned}$$

Hence  $\text{Kl}_{ta}^{n,m}(\psi; k_E/k_F) \neq \text{Kl}_t^{n,m}(\psi; k_E/k_F)$  for some  $t \in k_F^\times$ .  $\square$

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